# BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY

A Historical and Critical Review of Mathematical Science

EDITED BY

F. N. COLE VIRGIL SNYDER J. W. YOUNG
ALEXANDER ZIWET D. E. SMITH
T. LEVI-CIVITA R. C. ARCHIBALD

VOL. XXIII

October 1916 to July 1917

PUBLISHED BY THE
AMERICAN MATHEMATICAL SOCIETY
Providence, Rhode Island
1917

Reprinted 1969

## CONTENTS

### Whole No. 251


| | PAGE |
|---|---|
| Current Tendencies of Mathematical Research. By Professor E. B. VAN VLECK | 1 |
| Graphical Method of Finding the Possible Sets of Independent Generators of an Abelian Group. By Professor G. A. MILLER | 14 |
| On the Developments in Bessel's Functions. By Professor C. N. MOORE | 18 |
| Second Note on Removable Singularities. By DR. W. E. MILNE | 27 |
| The Mittag-Leffler Testament and Institute. Translated by DR. CAROLINE E. SEELY | 31 |
| The Madison Colloquium Lectures on Mathematics. By Professor G. A. BLISS | 35 |
| Shorter Notice | 44 |
| Notes | 46 |
| New Publications | 54 |


### Whole No. 252


| | PAGE |
|---|---|
| The Twenty-Third Summer Meeting of the American Mathematical Society. By Professor F. N. COLE | 57 |
| The Cambridge Colloquium. By Professor VIRGIL SNYDER | 81 |
| Note on the Order of Continuity of Functions of Lines. By Dr. C. A. FISCHER | 88 |
| The Equation of a Plane Rational Curve Defined by Parametric Equations. By Mr. H. W. RICHMOND | 90 |
| Scientific Method in Philosophy. By Professor C. J. KEYSER | 91 |
| Shorter Notices | 97 |
| Notes | 103 |
| New Publications | 106 |

## Whole No. 253


PAGE

An Extension of the Theory of Numbers by Means of Correspondences between Fields. By Professor L. E. DICKSON 109

Note on the Distribution of Quadratic Residues. By Mr. H. S. VANDIVER 111

Proof of a General Theorem on the Linear Dependence of $p$ Analytic Functions of a Single Variable. By Mr. H. M. MORSE 114

Note on the Linear Dependence of Analytic Functions. By Dr. G. A. PFEIFFER 117

On the Linear Dependence of Functions of One Variable. By Dr. G. M. GREEN 118

Translation Surfaces Associated with Line Congruences. By Professor O. E. GLENN 122

Pappus Introductory Paper. By Dr. J. H. WEAVER 127

Shorter Notices 136

Mathematics at an Italian Technical School. By Professor VIRGIL SNYDER 149

Notes 152

New Publications 154

## Whole No. 254


PAGE

The October Meeting of the American Mathematical Society. By Professor F. N. COLE 157

Correction 163

The Maximum Number of Cusps of an Algebraic Plane Curve, and Enumeration of Self-Dual Curves. By Professor M. W. HASKELL 164

Note on Asymptotic Expressions in the Theory of Linear Differential Equations. By Professor W. E. MILNE 166

On Notational Equivalence. By Professor E. B. WILSON 169

On Pierpont's Integral. Reply to Professor Pierpont. By Professor MAURICE FRÉCHET 172

A Reply to a Reply. By Professor JAMES PIERPONT 174

Shorter Notices 176

Notes 196

New Publications 199


## Whole No. 255


PAGE

The Twenty-Eighth Regular Meeting of the San Francisco Section. By Professor W. A. MANNING 203

The Tenth Regular Meeting of the Southwestern Section. By Professor O. D. KELLOGG 208

The Relations of Mathematics to the Natural Sciences. Presidential Address. By Professor E. W. BROWN 213

A Theorem on the Curves Described by a Spherical Pendulum. By Professor ARNOLD EMCH 230

A Theorem Concerning Continuous Curves. By Professor R. L. MOORE 233

A Note on Discontinuous Solutions in the Calculus of Variations. By Dr. P. R. RIDER 237

Theory of the Top. By Professor E. B. WILSON 241

Notes 244

New Publications 247

## Whole No. 256


PAGE

The December Meeting of the American Mathematical Society at Chicago. By Professor ARNOLD EMCH 251

The Twenty-Third Annual Meeting of the American Mathematical Society. By Professor F. N. COLE 257

Complete Existential Theory of the Postulates for Serial Order. By Professor E. V. HUNTINGTON 276

Complete Existential Theory of the Postulates for Well Ordered Sets. By Professor E. V. HUNTINGTON 280

Groups Generated by Two Operators of the Same Prime Order Such that the Conjugates of One under the Powers of the Other are Commutative. By Professor G. A. MILLER 283

A Theorem in the Analysis of Real Variables. By Professor R. L. BORGER 287

Concerning the Complement of a Countable Infinity of Point Sets of a Certain Type. By Dr. J. R. KLINE 290

An Analogue to Pascal's Theorem. By Dr. A. L. MILLER 292

Notes 293

New Publications 295


## Whole No. 257


PAGE

The February Meeting of the American Mathematical Society. By Professor F. N. COLE 299

The Equation of a Rational Plane Curve Derived from its Parametric Equations (Second Paper). By Professor J. E. Rowe 304

Ford's Studies on Divergent Series and Summability. By Professor C. N. MOORE 308

Shorter Notices 315

Notes 333

New Publications 337

## Whole No. 258


PAGE

Equilong Invariants and Convergence Proofs. By Professor EDWARD KASNER 341

The Inversion of an Analytic Function. By Dr. SAMUEL BEATTY 347

Emory McClintock 353

On Foci Conics. By Dr. J. H. WEAVER 357

New Light on our Numerals. By Mr. JEKUTHIAL GINSBURG 366

Shorter Notices 370

Notes 382

New Publications 384


## Whole No. 259


PAGE

The April Meeting of the American Mathematical Society at Chicago. By Professor ARNOLD DRESDEN 387

The April Meeting of the San Francisco Section. By Professor W. A. MANNING 401

Singular Points of Analytic Transformations. By Professor W. F. OSGOOD 404

The Projection of a Line Section upon the Rational Plane Cubic Curve. By Professor J. E. ROWE 405

Examples of a Remarkable Class of Series. By Professor R. D. CARMICHAEL 407

Shorter Notice 425

Notes 426

New Publications 431

## Whole No. 260


| | PAGE |
|---|---|
| The April Meeting of the American Mathematical Society in New York. By Professor F. N. COLE | 435 |
| The Modular Difference of Classes. By Professor P. J. DANIELL | 446 |
| On the Principal Units of an Algebraic Domain $k(\mathfrak{p}, a)$. By Dr. G. E. WAHLIN | 450 |
| Projective Geometry. By Professor J. L. COOLIDGE | 456 |
| Non-Euclidean Geometry. By Professor J. L. COOLIDGE | 461 |
| Correction | 467 |
| Notes | 467 |
| New Publications | 469 |
| Twenty-Sixth Annual List of Published Papers | 472 |
| Index to Volume XXIII | 482 |

BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY.

# CURRENT TENDENCIES OF MATHEMATICAL RESEARCH.*

BY PROFESSOR EDWARD B. VAN VLECK.

In an address which I had the pleasure of giving at the Atlanta meeting of the American Association for the Advancement of Science, I traced the influence of Fourier's series upon the development of mathematics. This instance is only one of many in which a central thought or problem has exerted a remarkable, sometimes even a controlling influence upon the development of mathematical thought. From early Greek times I may cite the historic problems of the duplication of the cube and the squaring of the circle. After the invention of analytic geometry it became essential to determine the tangent at a point of a given curve, and this led, as you know, to the discovery of the differential calculus, one of the most wonderful instruments ever devised for scientific thought. Almost simultaneously the problem of finding the area of a curve resulted in the construction of the definite integral, a concept even wider in scope than the differential coefficient. Still earlier, and near the beginning of the mathematical renaissance, the problem afforded by the equation of the $n$th degree was brilliantly solved for equations of the third and fourth degrees, and the first creative impulse subsequent to Greek times was thereby given to mathematics. In the last half of the nineteenth century this same problem

* This address was read at the Quarter Centennial of the University of Chicago before a conference of the mathematical, physical, and astronomical departments.

eventuated in the theory of groups which has become the foundation of mathematical classification.*

Examples such as these prompt us to ask: What are the central problems in the mathematical research of to-day? Each mathematical period has its own characteristics. The discovery of the calculus was followed by a notable period which I shall term the differential and gravitational epoch of mathematics, in which the consequences of the new calculus absorbed the attention, while at the same time it was developed as the handmaid of astronomy and physical science. The names of Euler on the one side, of Lagrange and Laplace on the other, serve sufficiently to specify the era and its two-fold aspects. For many reasons the last quarter of the nineteenth century might be termed the *group period*, from which we have only in part emerged. I need not tell you how necessary it became to explore geometry and analysis under the guidance of the group concept. Yet there are reservations which must be made in designating this quarter century as the group period. Likewise it is necessary to qualify any answer to the difficult question: What is the dominant problem or central thought in the research of to-day, if there be one? Nevertheless, let us be so importunate as to insist upon an answer, however foolhardy.

In the field of applied mathematics probably the "problem of three bodies" can be picked out as par excellence the present-day problem. Already in the hands of Poincaré this has given a first quickening of the fossilized methods of mathematical astronomy. In pure mathematics, to which I shall confine my attention, the number of conspicuous problems is legion, but above them all there looms, I think, in manifold aspects the problem of the *infinite set*. Analysis, geometry, and mechanics alike have been rapidly and increasingly permeated by the point set theory of Georg Cantor, in which the central core is the problem of the infinite set. More and more do the problems in these branches require, in some form or other, up-to-date knowledge of the infinite set. Consider, for example, what is called the problem of the primitive function. Given a function $F(x)$, what conditions must be imposed

* Many kinds of geometries and number fields have come to be distinguished. Their study and correlation are a feature of modern mathematical research which I have not found it convenient to discuss in this address.

upon it in order that there shall exist another or "primitive function" whose derivative is the given function? Or take the alternative question: What generalization shall be made in the notion of a derivative in order that as wide a class of functions as possible shall possess a primitive? Answers to these questions must involve a careful study of the infinite set of discontinuities of a function.

The characteristic tendency in the thought of to-day which I have tried to grasp under the comprehensive term "Problem of the infinite set" is shown rather as a current beneath the surface than in any individual concrete problem. The average investigator must perforce seize upon any problem which his brains find at hand. There exist, however, certain fundamental principles which will aid him in finding a worthy one. The great mathematician Jacobi is said to have inculcated upon his students the dictum: Man muss immer umkehren. One must always seek a converse, turn a thought the other end to. It was by turning the elliptic integral inside out that Jacobi obtained his splendid theory of elliptic and theta functions. As other instances of the same principle I cite at random the implicit function theory which underlies so much modern investigation, and the perception of the existence of transcendental numbers through failure of the finite processes which produce our ordinary numbers.

Without dwelling further upon the fertility of Jacobi's dictum, I wish to coin and put beside it another obvious dictum of yet wider reach: Man muss immer generalizieren. By this I do not mean cheap generalization to $n$ dimensions or variables of that which has been already done for two or three. Not infrequently, however, generalization even to $n$ variables is a problem of importance and difficulty, inasmuch as the solution for the special case of two or three variables may possess a distinctive structure and character and hence be in no way typical. A notable instance of such difficulty is found in the generalization of the ordinary second order conditions for the maximum of a definite integral in the calculus of variations, while extraordinary obstacles are encountered in the extension of the theory of analytic functions to two or more variables. But whether or not extension to $n$ dimensions be trivial, generalization to a countably infinite number of dimensions becomes sublime! With a further extension to an uncountably infinite number of dimensions or variables

it might seem that we would reach a sublimated or etherial stage of mathematical development, but such flippant characterization would give an altogether false impression of Volterra's theory of functions of lines.

Extension to a countably infinite number of variables commonly affords opportunity for the finest sagacity and insight. An excellent program for work could be found in extension of almost any finite theory. It is your own Professor Moore whom I have heard glowingly preach that to every finite theory there must correspond, under proper limitations, a general transcendental theory with an infinite number of variables. What is more beautiful and simple in elementary mathematics than the application of the theory of determinants to the solution of a set of $n$ linear equations in $n$ variables

$$(1 + a_{11})x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = C_1,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{1n}x_1 + a_{2n}x_2 + \cdots + (a_{nn} + 1)x_n = C_n;$$

such a system as we learned to solve even before we reached quadratic equations in algebra? Now it was precisely a passage from this simple system of linear equations to a system with a countably infinite number of unknowns which gave to Volterra and Fredholm the basis of their theory of integral equations. At the limit the system of linear equations passes into a linear integral equation

$$u(x) + \int_a^b u(\xi)K(x, \xi)d\xi = f(x),$$

in which everything is given except the function $u(x)$ which is to be found. The solution $(x_1, x_2, \cdots, x_n)$ of the linear system with the indefinite increase of $n$ passes over into the solution $u(x)$ of the integral equation, but the legitimacy of the passage to the limit and of the solution must be verified in some manner. From the very outset the theory of integral equations has been found very useful in applications to mathematical physics and elsewhere. This might be expected because it is the generalization of the elemental theory of a set of linear equations.

The solution of integral equations is only one of many evidences of the mathematician's tightening grip upon the

infinite. The infinite increase in the number of variables in our mathematical problems necessitates geometrically a theory of space of countably infinite dimensions. Already a promising beginning has been made by restricting the attention to Hilbertian space—that is, to that portion of infinitely-dimensioned space for which $\Sigma x_n^2$, the sum of the squares of the coordinates of a point, is convergent. The familiar conditions for orthogonality of lines, etc., then generalize without material modification.

The theory of a space of infinitely many variables is companion to an incipient theory of functions of an infinite number of variables. I fancy a smile, inward or outward, on the faces of some of the physicists and astronomers present, to whom such a function theory may seem highly typical of the theoretic, up-in-the-air character of the mathematician and his work. Let me try to address to you some arguments ad hominem. Have you nothing of such character in physics? In my early life, when I was hesitating between mathematics and physics as a profession, I first encountered Lagrange's generalized coordinates, whose number, by the way, need not be finite. The equations for motion, energy, etc., were set up in terms of these generalized coordinates. I shall never forget how the glittering generality of the theory astonished and fascinated me. And when the equations were actually applied to sundry objects—as, for instance, to the gyroscope —it seemed to me then, as it almost does still, about the finest example of getting something intellectually out of nothing that I had ever come across. The very generality of the coordinates gives to them adaptability to the varying complications of physical science. Now most phenomena of nature depend in their final analysis upon an infinite number of variables. How then can it be thought that we can master them unless the mathematician prepares the way by developing a theory of functions of infinitely many variables? The potential or force due to an electric current in a wire is a function of the shape of the wire and depends therefore upon the uncountably infinite number of its points. Indeed, it is not the pure mathematician who has been so hardy and rash as to attempt the beginning of a theory of functions of lines. No, the bold pioneer in this new field is no other than a mathematical physicist or, as we may claim with equa truth, a physical mathematician—Volterra, a superb interpreter of current tendencies.

Time fails me to speak of the closely related and fundamental "functional calculus" initiated by Fréchet, certain aspects of which are being developed here at Chicago by Moore and his pupils.

An effect, and likewise a cause, of the generalized aims in mathematics, of which I have thus far spoken, has been the creation of refined mathematical tools. Under the term "mathematical tool" I mean to include concepts and algorisms which are usable for the demonstration of mathematical truth. The importance of concepts broadly comprehensive in their reach is shown alike by the history of geometry and of analysis. The open plane of Euclid yields a general geometry only after the introduction of ideal elements which we call the point, line, etc., at infinity. The creation thereby of a closed and perfect projective geometry was one of the milestones of mathematical progress in the century just past. Poncelet's comprehensive principle of geometric continuity, justified at first rather by necessity than by adequate logic, was another indispensable geometric creation. In the notion of a group we have a concept which reaches pervasively through the realms of geometry and analysis. Still another example of the fecundity of the subtler concepts forged by the progress of science is to be found in Hill's "periodic orbit," introduced in the theory of attraction as a norm from which to reckon the deviation of the orbit of the heavenly body.

My remaining illustrations will be taken chiefly from analysis, partly because it is the province with which I am most familiar, but also because the same general tendency which was exhibited conspicuously in the magnificent enlargement of geometry in the half century from 1820 to 1870 is to-day manifested in the broadening of analysis. Consider the familiar notion of an integral, apparently moulded to a finality in the hands of Riemann. This admits a finite number of discontinuities but an infinite number only under certain narrow restrictions. A totally discontinuous function—for example, one equal to zero in the rational points which are everywhere dense in the interval of integration, and equal to 1 in the irrational points which are likewise everywhere dense—is not integrable à la Riemann. The restriction became a very hampering one when mathematicians began to realize that the analytic world in which theorems are deducible does not consist merely of highly civilized and

continuous functions. In 1902 Lebesgue with great penetration framed a new integral which is identical with the integral of Riemann when the latter is applicable but is immensely more comprehensive. It will, for instance, include the totally discontinuous function above mentioned. This new integral of Lebesgue is proving itself a wonderful tool. I might compare it to a modern Krupp gun, so easily does it penetrate barriers which before were impregnable.

Some of the newer mathematical concepts resemble the modern telescopes, disclosing new worlds of theorems and ideas bound together in a common system. Other concepts because of their high dispersive power act more like the optical grating, making new distinctions visible. This is particularly true of the modern theory of point sets which gives a grip on weird sets of points almost inconceivable in their irregularity. Take for consideration the two sets of points of which I spoke a few moments ago, the set of rational points everywhere dense and the set of irrational points likewise everywhere dense in a given interval, confusedly mixed together like grains of pepper and salt. Which predominates? Should the mixture be called white or black? Would it seem possible to say? Yet the theory of point sets tells us that the black—i. e., the irrational points—predominate; that the rational points are the exception and have a measure 0, while the irrational points have a measure equal to the length of the interval. The rational points are weighed in the balance by Borel and Lebesgue and are found altogether lacking, at least in measure. They afford an example of what may be termed a null set, that is, a set of measure 0. It is natural to expect that such a set of points would be without influence on many results—for example, on the value of a Lebesgue integral. This turns out to be the case. By allowing for a set of exceptional points of measure 0 many theorems can now be perceived and demonstrated which before were unthinkable.

Along with the widened aims and concepts of mathematics have come fresh methods and distinctive styles of thought. Look first at the older methods. The work of the grand old mathematicians of the seventeenth and eighteenth centuries was based on definite algorisms, such as the hypergeometric series of Gauss and the Taylor power series. Euler, in particular, was astonishingly prolific in algorisms. When the

function theory sprang into existence it became even more indispensable to make elaborate studies of special algorisms and functions. A host of particular functions absorbed the attention of mathematicians—elliptic functions, automorphic functions, Bessel's functions, other functions defined by differential equations, and so on. The function theory of Weierstrass was based on a definite algorism, the Taylor power series. In recent times other processes and functions have received attention, for example, algebraic continued fractions and divergent series, factorial series, Dirichlet series, and so on. Definite algorismic work must, indeed, always remain of primary importance, not only on account of its use in calculation but also because the progress of our science continually introduces new functions for study. At the same time older processes need to be reexamined under modern lights,—for instance, the difference equation which has been considered by Nörlund, Birkhoff, and Carmichael in the light of the analytic function theory.

In sharp contrast with this algorismic work so characteristic of earlier time is a distinctively modern turn of thought which perhaps first found a clear exponent in Riemann. He presents a strange antithesis to his contemporary countryman, Weierstrass. Riemann bases the function theory upon a property rather than upon an algorism—to wit, the possession of a differential coefficient by the function in the complex plane. Thus at a stroke it is freed from dependence upon a particular process like the power series of Taylor. His celebrated memoir upon the *P*-function is a characteristic development of a whole Schar (family) of functions from their mutual relations.

A first aspect of the peculiar thought-mode of which Riemann's is a type is definition and development of a function from its properties. One illustration is found in the definition of a function by building a given region point for point and conformally upon another given region. In severe cases the regions may even contain an infinite number of leaves properly bound together. This illustration is significant also of a certain modern tendency to build mathematics upon correspondence and order. The boundary value problems of mathematical physics afford another example of definition of a function through its properties. As an effect of this tendency we may expect ultimately a great develop-

ment of the incipient theory of functional equations. In this theory a single equation or system of equations expressing some property is taken as the definition of a class of functions whose characteristics, particular as well as collective, are to be developed as an outcome of the equations.

A second phase of the same thought tendency is to be found in the existence theorems so characteristic of modern analysis. Existence proofs are, indeed, a preliminary and indispensable element, if functions are to be defined through their properties. Their rôle has been discussed so often that I need not enlarge upon it.

A third phase of the same tendency is the modern examination of the postulates of geometry, arithmetic, algebra, and even mechanics. This postulational work seizes the material at the root before development sets in. It has to face the obvious objection that it is an exercise in sterile logic rather than creative mathematics. A swirling brain like Poincaré's will never stop in its constructive work for abstract postulational considerations, but will leap gaps like an electric spark. To this objection the rejoinder may be made that critical revision is not so far from creative thought as is generally supposed; that postulational theories are to-day much more fertile than in the past, that such postulational developments as Hilbert's in his famous Foundations of Geometry and his foundations for a geometric group theory are, in fact, creative work of highest order; that Lebesgue formulated his concept of a generalized integral in the light of postulational requirements; and so on.

In this connection the primal thought which lies at the bottom of Moore's General Analysis deserves special consideration. When, he argues, a number of similar results in different fields are compared, the belief is forced irresistibly upon one that the essential oneness of the results must be due to identity of the hypotheses, this identity being partially or altogether veiled from sight by the admixture of foreign ingredients, as in chemical compounds. To apply a technical term, we have before us theories to be rendered simply isomorphic. The problem before the mathematician is accordingly to extract the common basis and exhibit the results as consequences of a common set of hypotheses applicable to varying conditions and objects. Thereby not only will like theories be coordinated, but the common underlying prin-

ciples laid bare will be ready for use elsewhere. Doubtless the mathematician will continue to be guided for the most part by analogy in the future as in the past, but I doubt not that the isolation of the common elements in like theories will be stimulating and productive, even as the postulational study of the foundation of geometry has led to the discovery of new geometries. The general analysis has furthermore the advantage over ordinary postulational investigations of not tapping the root but the sap.

The change in method of which I have spoken centers largely in the prescription of properties and postulates rather than of algorisms. Entirely different from this is the increasing tendency to recast a problem so as to make it fertile. In the words which I have heard Hadamard employ, a problem must be "bien posé," fittingly formulated in order to give rich result. The inner nature of the problem needs to be comprehended. It carries in itself its own inherent limitations on its solution. To illustrate by drawing upon the calculus of variations, the problem of minimizing the integral

$$\int F(x, y, y', y'', \cdots, y^{(n)})dx$$

by a function $f(x)$ is badly formulated and does not admit of precise answer unless we specify the class of functions from which the solution is to be taken. If we take, for example, the solution from functions of class $C^{(n)}$ in Bolza's terminology —that is, functions which are continuous together with their derivatives down to the $n$th inclusive—the problem then becomes well formulated and can be tackled successfully. In this particular case we impose limitations in order to well-formulate the problem. But it is distinctly modern to make a problem solvable by widening it so as to make it more instead of less comprehensive. Thus a problem not possible of general solution if only functions integrable in the ordinary Riemannian sense are considered, may become so if Lebesgue integrals are admitted. In fact, the solution may depend upon the kind of tool employed, and a careful study of the adaptation of the tool to the problem and of the problem to the tool should therefore be made. Thus if the problem should involve convergence, its character and formulation may change according as the tool involved is ordinary convergence, uniform convergence, quasi-uniform

convergence, convergence "en moyen," or essentially uniform convergence.

In conclusion, partly by way of illustration and partly by way of summary, let me present a theorem which embodies most of the tendencies of which I have spoken. This is the so-called Fischer-Riesz theorem, remarkable for its elegant character. You all know the representation of a function $f(x)$ by a Fourier series,

$$\frac{a_0}{2}+\frac{1}{\pi}\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx),$$

in which the coefficients are obtained from the function by the equations

$$a_n=\int_{-\pi}^{\pi}f(x)\cos nx\,dx, \qquad b_n=\int_{-\pi}^{\pi}f(x)\sin nx\,dx.$$

Now as Jacobi said, Man muss immer umkehren. Instead of representing a known function by a Fourier series, let us seek conversely to represent a given Fourier series by a function. The given series may be either convergent or divergent. We have now before us one of the great problems of analysis.

But before considering the problem let us obey the injunction: Man muss immer generalizieren. The Fourier series is only one of a class of series

$$\text{(I)} \qquad a_1\phi_1(x)+a_2\phi_2(x)+a_3\phi_3(x)+\cdots,$$

in which the functions have with respect to a given interval $(a, b)$ the so-called orthogonality property,

$$\int_a^b\phi_i(x)\phi_j(x)dx=0 \qquad (i\neq j).$$

If, further, the functions have been normalized, as was the case above in the Fourier series, we have also

$$\int_a^b\phi_i^2(x)dx=1.$$

Such a series is called a *generalized Fourier's series*, of which many special cases are well known, as, for instance, a series in terms of Legendre polynomials.

The problem now before us is to represent such a series, whether convergent or divergent, by a function and to derive

the function from the series. We shall agree to say that a function $f(x)$ represents the series if 1) it is connected with the series by the equations

$$a_n = \int_a^b \phi_n(x) f(x) dx$$

and thus generates the series in the customary way; and if 2) the function agrees in value with the series at all points of the interval $(a, b)$ for which the series is convergent. The questions now confront us: When will such a function exist? How can it be obtained from the series? To what extent is it unique? In this shape the problem would be almost or quite insoluble. The problem is not yet "bien posé." For good formulation we must change the tool and use Lebesgue integrals throughout, thus extending the class of functions admissible for the representation of the series. Furthermore, in deriving the function from the series we will not require that the function and the series shall agree at all points of convergence but will allow exceptions at an infinite number of points forming a set of measure 0.

It is now possible to state the theorem which Fischer and Riesz obtained independently by different methods. The necessary and sufficient condition that there shall exist a function representing the series (and having an integrable square $f^2(x)$) is that the sum of the square of the coefficients, $\Sigma a_n^2$, shall be convergent; in other words, the point $(a_1, a_2, \cdots)$ must lie within Hilbertian space of infinitely many dimensions. The function is completely determined save at an arbitrary set of points of measure zero, where we can give the function any value we choose. It can be obtained from the series by a proper grouping of the terms without change of order, which will render the series convergent except possibly at a set of points of measure 0. The study of the convergence of the series has brought to light two new modes of convergence, one of which, Weyl's essentially uniform convergence, seems to me very important. The other I shall stop to explain briefly because it is an admirable example of generalized concept and because I think that it may interest those of you who are acquainted with the theory of least squares. If $S_n(x)$ denotes the sum of $n$ terms of our generalized Fourier series (I) and if $f(x)$ is, as before, the function represented by the series, the square of the error at any

point $x$ of the interval is $[S_n(x) - f(x)]^2$, and the sum of the squares of the errors taken over the interval is the integral $\int_a^b [S_n(x) - f(x)]^2 dx$. As $n$ increases indefinitely, this error integral approaches 0, so that the series converges "en moyen" (on the average) although there may be no true convergence in the ordinary sense of the term. Conversely, if

$$\int_a^b [S_{n+m}(x) - S_n(x)]^2 dx$$

can be made as small as we please by taking $n$ sufficiently great, then there exists a function $f(x)$ toward which the series converges en moyen.

But enough of this. I trust that you see that the Fischer-Riesz theorem affords an elegant example of modern mathematical research—characteristic in its generalized sweep, in its creation and use of refined tools and concepts, in its recasting of the problem so as to make it fertile, and lastly in its attainment of success through point set considerations, involving in this case the recognition of a null set of exceptional points. Not a little of the work of analysts for some time to come must center around the numberless new notions until their "Tragweite"—their carrying power and reach—is determined. To a great extent they have modified the character of current investigation. There is regretably less of the action and interaction of mathematics and mother nature. There is also insufficient study of approximation and astronomical mathematics—fields which are fortunately represented here by Moulton and MacMillan and which serve to counteract or supplement a somewhat widespread abstruse inclination in our country. But the newer thought does make tremendous advance when questions of theory rather than of calculation are involved, and the combination of sweeping generalization with rigor is astonishing. There remains yet to be accomplished the complete interweaving and correlating of the old and the new. We cannot see far into the future, but the well of mathematics will not run dry as long as there remains such springs as the problem of three or $n$ bodies, and the tantalizing, old, yet unsolved problem of the shape of curves of the $n$th degree.

UNIVERSITY OF WISCONSIN,
*June*, 1916.

# GRAPHICAL METHOD OF FINDING THE POSSIBLE SETS OF INDEPENDENT GENERATORS OF AN ABELIAN GROUP.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, April 22, 1916.)

IF the order of an abelian group $G$ is not a power of a prime number, a set of independent generators of $G$ can be obtained by taking an arbitrary set of independent generators of each of the Sylow subgroups of $G$ and regarding the totality of operators in the sets thus obtained as a single set. This is true irrespective of which of the two common definitions of a set of independent generators may be adopted. According to one of these two definitions any set of $l$ operators of $G$ which generate $G$ and are such that no $l-1$ of them generate $G$ is said to be a set of independent generators of $G$. This is sometimes called the general definition of a set of independent generators, and every possible group has at least one such set of independent generators. According to the restricted definition of a set of $l$ independent generators, the additional condition is imposed that the group generated by every $l-1$ of these operators has only identity in common with the group generated by the remaining one.

It is evident that not every possible group has a set of independent generators according to the restricted definition. For instance, the quaternion group does not have any such set of independent generators. It is well known that every possible abelian group has at least one set of independent generators according to each of these two definitions, and that the number of the operators in such a set is the same under both definitions whenever the group is of prime-power order and has at least one set of independent generators satisfying both definitions. The latter statement remains true even when the group of prime-power order is non-abelian.

Since the determination of a set of independent generators of an abelian group whose order is not a power of some prime number may be made to depend upon the determination of such a set of independent generators for each of its Sylow subgroups, irrespective of which of the two given definitions

of a set of independent generators is adopted, we shall confine ourselves, in what follows, to the consideration of the possible sets of independent generators of an abelian group $G$ whose order is $p^m$, $p$ being a prime number. For brevity it will always be assumed, in what follows, that the independent generators under consideration satisfy the restricted definition unless the contrary is explicitly stated.

Let $H$ be the characteristic subgroup of $G$ composed of the operators obtained by raising each operator of $G$ to its $p$th power. The quotient group $G/H$ is of type $(1, 1, 1, \cdots)$, and if the order of this quotient group is $p^k$, $G$ has exactly $k$ independent generators. Moreover, whenever $G$ has exactly $k$ independent generators the order of this quotient group is $p^k$. None of the operators of $H$ can be used as an independent generator of $G$, irrespective of which of the two definitions of independent generators is adopted. According to the general definition of independent generators, $H$ is composed of all the operators of $G$ which cannot appear in some one of the possible sets of independent generators of $G$; that is, $H$ is the $\phi$-subgroup of $G$.

A necessary and sufficient condition that every operator of $G$ which does not occur in $H$ appears in at least one of the possible sets of independent generators of $G$ is that no two invariants of $G$ have a quotient which exceeds $p$, and a necessary and sufficient condition that all the operators of $G$ which are not contained in $H$ are conjugate under the group of isomorphisms of $G$ is that all the invariants of $G$ are equal to each other. When the latter condition is satisfied a set of independent generators of $G$ can be chosen in

$$p^{mk-k^2}(p^k - 1)(p^k - p)(p^k - p^2) \cdots (p^k - p^{k-1})$$

different ways. The first of the factors in this product represents the number of ways in which these independent generators can be selected after the co-sets from which they are to be taken have been determined, while the product of the other factors represents the number of ways in which these co-sets can be chosen.

Since the order of the group of isomorphisms of any abelian group is equal to the number of different ways in which a set of independent generators of the group can be selected, it results that the above formula represents the order of the group of isomorphisms of $G$ whenever all the $k$ invariants

of $G$ are equal to each other and only then. The same formula clearly represents the number of ways in which a set of general independent generators of $G$ can be chosen irrespective of whether the $k$ invariants of $G$ are equal to each other or do not have this property.

Suppose that a set of independent generators of $G$ contains $k_1$ operators of order $p^{\alpha_1}$, $k_2$ of order $p^{\alpha_2}$, $\cdots$, $k_\lambda$ of order $p^{\alpha_\lambda}$, $\alpha_1 < \alpha_2 < \cdots < \alpha_\lambda$. Then $k_1 + k_2 + \cdots + k_\lambda = k$, and $G$ contains exactly $p^{k_1} - 1$ co-sets with respect to $H$ which involve operators of order $p^{\alpha_1}$ but no operators of lower order. These co-sets together with $H$ constitute a subgroup $H_1$ of $G$ which includes all the operators of $G$ whose orders do not exceed $p^{\alpha_1}$. The co-sets which involve operators of order $p^{\alpha_2}$, but no operators of lower order, together with $H_1$ constitute a subgroup $H_2$ of $G$ which includes all the operators of $G$ whose orders do not exceed $p^{\alpha_2}$, etc.

*A necessary and sufficient condition that any given operator of $G$ appears in at least one set of independent generators is that it belongs to a co-set of $G$ with respect to $H$ and that it is one of the operators of lowest order in this co-set.* If such an operator is of order $p^\alpha$, the co-set in which it occurs has as many operators of order $p^\alpha$ as there are operators in $H$ whose orders divide $p^\alpha$, $\alpha_1 \leqq \alpha \leqq \alpha_\lambda$. Hence the following diagram

| $H$ | | Co-sets of $G$ as regards $H$ whose smallest operators are of order | | | |
|---|---|---|---|---|---|
| | | $p^{\alpha_1}$ | $p^{\alpha_2}$ | . . . . . | $p^{\alpha_\lambda}$ |
| Operators of $H$ whose orders divide | $p^{\alpha_1}$ | Possible independent generators of order $p^{\alpha_1}$ | Possible independent generators of order $p^{\alpha_2}$ | | |
| | $p^{\alpha_2}$ | | | | |
| | . . . . . | | | . . . . . | Possible independent generators of order $p^{\alpha_\lambda}$ |
| | $p^{\alpha_\lambda - 1}$ | | | | |

may serve to exhibit the operators from which a set of independent generators must be selected:

This diagram exhibits not only the operators from which the independent generators of different orders must be selected and the operators which cannot be used as such generators, but also the number of ways in which the independent generators of any possible order can be chosen. For instance, if $G$ contains independent generators of order $p^{\alpha_\gamma}$ and if $p^{\alpha'_\gamma}$ is the order of the subgroup of $H$ which is composed of all its operators whose orders divide $p^{\alpha_\gamma}$, then the $k_\gamma$ generators of order $p^{\alpha_\gamma}$ can be chosen in a number of ways represented by the following product:

$$p^{(\alpha'_\gamma+k_1+k_2+\cdots+k_{\gamma-1})k_\gamma}(p^{k_\gamma}-1)(p^{k_\gamma}-p)\cdots(p^{k_\gamma}-p^{k_\gamma-1}),$$

where $k_{\gamma-1}=0$ when $\gamma=1$. The order of the group of isomorphisms of $G$ is therefore equal to

$$\prod_{\gamma=1}^{\gamma=\lambda} p^{(\alpha'_\gamma+k_1+k_2+\cdots+k_{\gamma-1})k_\gamma}(p^{k_\gamma}-1)(p^{k_\gamma}-p)\cdots(p^{k_\gamma}-p^{k_\gamma-1}).$$

Instead of considering the co-sets of $G$ with respect to $H$ we may consider the co-sets of $H_1$ with respect to $H$, then those of $H_2$ with respect to $H_1$, etc. Finally, those of $G$ with respect to $H_{\lambda-1}$ are considered for the purpose of finding the possible independent generators of highest order. According to this method, the determination of the possible sets of independent generators of $G$ is simply a repetition of the process employed to determine the independent generators of $G$ which are of order $p^{\alpha_1}$ and are contained in $H_1$. This method of determining the possible sets of independent generators of $G$ has contact with the method described by the writer in the *Tôhoku Mathematical Journal*, volume 2, 1912, page 137. Its main advantage is due to the fact that it exhibits clearly the operators which can appear in some possible set of independent generators.

UNIVERSITY OF ILLINOIS.

# ON THE DEVELOPMENTS IN BESSEL'S FUNCTIONS.

BY PROFESSOR CHARLES N. MOORE.

(Read before the American Mathematical Society, January 1, 1916.)

IN studying the developments in Bessel's functions it is found that the behavior of the series at the origin and in a neighborhood including the origin is, in certain cases, difficult to determine. For example most of the proofs of the convergence of these developments leave unsettled the question of the convergence at the origin of the developments in Bessel's functions of order zero. Also, most of the discussions of the uniform convergence of these developments leave unsettled the question of the uniform convergence of the developments in a neighborhood including the origin. Yet for many of the applications to mathematical physics it is essential to settle both of these points in order to be sure that we have a solution of the physical problem we are discussing.

It was a consideration of these facts that led the writer several years ago to devise a method of establishing the convergence of the developments in Bessel's functions that would settle both of these questions under conditions sufficiently wide for the applications.* The method referred to, however, serves only to establish the convergence or uniform convergence of the developments, and it is necessary to determine the value to which they converge by means of other considerations. In the paper just referred to this was done by using the results of previous writers in connection with the value of the developments. As these results are obtained by means of long and complicated discussions, it seems highly desirable to have some fairly simple method of determining the value of the development when we know it converges. Such a method is given in the present paper, and consequently this paper combined with the previous one gives the first *complete* discussion of the convergence and value of the developments in Bessel's functions under conditions that are usually satisfied in the applications.

* Cf. *Transactions*, vol. 12 (1911), p. 181.

The method used here was first suggested by Sturm and Liouville in connection with the Sturm-Liouville expansions, and the application of it to these expansions was sketched out by them.* Some forty years later Heine attempted to apply this method to the developments in Bessel's functions.† His treatment of this case is inadequate, however, for by omitting practically all the details he avoids all the difficulties. Moreover, it is apparent that he did not himself examine the details of the proof with sufficient care, for he fails to note the presence of an extra term in the development in certain exceptional cases, a fact first observed by Dini.‡

The cases just mentioned are not essentially exceptional but only appear so on account of the notation ordinarily employed. As pointed out before,§ they can be included in the general treatment by means of adopting the following notation. We set

$$(1) \qquad F_\nu(\lambda, x) \equiv \left(\frac{x}{2}\right)^\nu \sum_{i=0}^{\infty} \frac{(-1)^i (\lambda x)^{2i}}{2^{2i} i!\, \Gamma(\nu + i + 1)} \equiv \frac{1}{\lambda^\nu} J_\nu(\lambda x)$$

and consider the series

$$(2) \qquad \sum_{n=1}^{\infty} F_\nu(\lambda_n, x) \frac{\int_0^1 x f(x) F_\nu(\lambda_n, x)\, dx}{\int_0^1 x [F_\nu(\lambda_n, x)]^2 dx} = \sum_{n=1}^{\infty} A_n F_\nu(\lambda_n, x),$$

where the $\lambda$'s are the successive roots, positive or zero of the equation

$$(3) \qquad \omega(\lambda) = \left[ l \frac{\partial}{\partial x} F_\nu(\lambda, x) + h F_\nu(\lambda, x) \right]_{x=1} = 0,$$

arranged in increasing order of magnitude.

The object of the present paper is to show that whenever the series (2) converges or is summable at all points of the interval $0 \leqq x \leqq 1$ at which $f(x)$ is continuous, then under certain conditions to be stated subsequently the value of the series at each of these points will be $f(x)$. We will begin by establishing some lemmas.

* Cf. *Liouville's Journal*, vol. 2 (1837), p. 220.
† Cf. *Crelle's Journal*, vol. 89 (1880), p. 35.
‡ Dini, Serie di Fourier, Pisa, 1880.
§ Cf. *Transactions*, vol. 10 (1909), p. 420, and vol. 12 (1911), p. 182.

LEMMA 1. *The integrals*

$$\frac{1}{2\pi i}\int_{R_n}\frac{zF_\nu(z,x)}{(z-\alpha)\omega(z)}dz,$$

$$\frac{1}{2\pi i}\int_{R_{n'}}\frac{F_\nu(z,x)-x^\nu F_\nu(z,1)}{(z-\alpha)z^2F_\nu(z,1)}dz \quad (\alpha \text{ a real constant}), \tag{4}$$

*where* $\omega(z)$ *is defined by* (3), $l$ *not being zero, and the path of integration is a rectangle in the complex plane with vertices at the points* $(k, \sqrt{k})$, $(-k, \sqrt{k})$, $(-k, -\sqrt{k})$ *and* $(k, -\sqrt{k})$, $k$ *being chosen equal to*

$$n\pi+\frac{2\nu+3}{4}\pi \quad \text{or} \quad n\pi+\frac{2\nu+1}{4}\pi \quad (n \text{ a positive integer}), \tag{5}$$

*according as we are considering the first integral or the second integral, will approach zero as* $n$ *becomes infinite, provided* $x$ *lies in the interval* $0<x<1$.

We shall give the proof in detail for the first only of the integrals in (4), the treatment of the second integral being analogous.

We have for $J_\nu(z)$ the following asymptotic expansion*

$$J_\nu(z)=\frac{1}{\sqrt{2\pi z}}[e^{i(z-\gamma)}(1+\theta_1(z,\nu))+e^{-i(z-\gamma)}(1+\theta_2(z,\nu))], \tag{6}$$

where $\gamma=\frac{1}{4}(2\nu+1)\pi$ and

$$|\theta_1(z,\nu)|<\frac{K_1}{|z|}, \quad |\theta_2(z,\nu)|<\frac{K_2}{|z|} \quad \begin{pmatrix}|z|>c>0\\ R(z)>0\end{pmatrix}, \tag{7}$$

$K_1$ and $K_2$ being positive constants for a fixed value of $\nu$.

Making use of (1) and the well-known relation

$$J_\nu'(z)=-J_{\nu+1}(z)+\frac{\nu}{z}J_\nu(z), \tag{8}$$

we obtain

* Cf. H. Weber, "Zur Theorie der Bessel'schen Functionen," *Math. Annalen*, vol. 37 (1890), p. 404. Equation (6) above results from a combination of equations (19), (30) and (33) of the paper referred to, and is established there for all values of $\nu>-\frac{1}{2}$, and for all values of $z$ whose real part is positive. It can be at once extended to values of $z$ along the axis of imaginaries, the origin being excepted, on account of the continuity of all the functions involved, and for values of $z$ to the left of the axis of imaginaries we can get a corresponding expression by means of the relationship $J_\nu(-z)=(-1)^\nu J_\nu(z)$.

$$\omega(z) = \frac{1}{z^\nu}[(l\nu + h)J_\nu(z) - lzJ_{\nu+1}(z)]. \tag{9}$$

For the sake of brevity we set

$$\theta_1 = \theta_1(\pm z, \nu), \quad \theta_1' = \theta_1(\pm zx, \nu), \quad \theta_2 = \theta_2(\pm z, \nu),$$

$$\theta_2' = \theta_2(\pm zx, \nu), \quad \theta_3 = \theta_1(\pm z, \nu + 1), \quad \theta_4 = \theta_2(\pm z, \nu + 1).$$

Then if we further set $z = p + qi$ and make use of (1), (6), the extension of (6) indicated in the previous footnote and (9), we see that the integrand of the first integral in (4) may be written in the form

$$\frac{z[e^{i(\pm px-\gamma)}e^{\mp qx}(1+\theta_1') + e^{-i(\pm pz-\gamma)}e^{\pm qx}(1+\theta_2')]}{\sqrt{x}(z-\alpha)[\mp lz\{e^{i[\pm p-\gamma-(\pi/2)]}e^{\mp q}(1+\theta_3) + e^{-i[\pm p-\gamma-(\pi/2)]}e^{\pm q}(1+\theta_4)\} + (l\nu+h) \times \{e^{i(\pm p-\gamma)}e^{\mp q}(1+\theta_1) + e^{-i(\pm p-\gamma)}e^{\pm q}(1+\theta_2)\}]}. \tag{10}$$

For points on the sides of the rectangle $R_n$ that are parallel to the $y$-axis, i. e., for values of $z$ of the form $\pm k + iq$, where $k$ has the first of the values (5), (10) reduces to

$$(-1)^n \frac{e^{\mp qx}e^{i(kx-\gamma)}(1+\theta_1') + e^{\pm qx}e^{-i(kx-\gamma)}(1+\theta_2')}{\mp l\sqrt{x}\{\pm((n+\frac{1}{2})\pi+\gamma)-\alpha+iq\}(e^q+e^{-q})(1+\theta_5)},$$

where for large values of $|z|$, $\theta_5$ is an infinitesimal of the same order as the other $\theta$'s. It is readily seen from this latter form that if $n$ is chosen large enough to make $\theta_1'$, $\theta_2'$ and $\theta_5$ each less in absolute value than $\frac{1}{2}$, the integrand in (4) will for the values of $z$ under consideration be less in absolute value than

$$\frac{6}{l\sqrt{x}\{(n+\frac{1}{2})\pi+\gamma-\alpha\}}.$$

Hence the contributions to the first integral in (4) from the integrations along the sides of the rectangle parallel to the $y$-axis, will be less in absolute value than

$$\frac{1}{2\pi} \cdot \frac{24\sqrt{(n+\frac{1}{2})\pi+\gamma}}{l\sqrt{x}\{(n+\frac{1}{2})\pi+\gamma-\alpha\}}, \tag{11}$$

and therefore will approach zero as $n$ becomes infinite.

For points on the sides of the rectangle $R_n$ that are parallel to the $x$-axis, i. e., for values of $z$ of the form $p \pm i\sqrt{k}$, $k$ having the same value as above, (10) reduces to*

$$\frac{e^{i(\pm pz-\gamma)}e^{\mp(\pm)\sqrt{k}z}(1+\theta_1') + e^{-i(\pm pz-\gamma)}e^{\pm(\pm)\sqrt{k}z}(1+\theta_2')}{\mp l\sqrt{x}\left(p-\alpha+(\pm)\,i\sqrt{(n+\tfrac{1}{2})\pi+\gamma}\right)e^{\sqrt{k}}e^{\mp(\pm)i[\pm p-\gamma-(\pi/2)]}(1+\theta_6)},$$

where for large values of $|z|$, $\theta_6$ is an infinitesimal of the same order as the other $\theta$'s. It is readily seen from this form that if $n$ is chosen large enough to make $\theta_1$, $\theta_2$, and $\theta_6$ each less in absolute value than $\frac{1}{2}$, the integrand of the first integral in (4) will for the values of $z$ under consideration be less in absolute value than†

$$\frac{6}{l\sqrt{x((n+\frac{1}{2})\pi+\gamma)}e^{(1-x)\sqrt{(n+\frac{1}{2})\pi+\gamma}}}.$$

Hence the contributions to the first integral in (4) from the integrations along the sides of the rectangle parallel to the $x$-axis, will be less in absolute value than

$$\frac{1}{2\pi}\cdot\frac{24\{(n+\frac{1}{2})\pi+\gamma\}}{l\sqrt{x((n+\frac{1}{2})\pi+\gamma)}e^{(1-x)\sqrt{(n+\frac{1}{2})\pi+\gamma}}}, \tag{12}$$

and therefore will approach zero as $n$ becomes infinite.

Combining the previous results it is seen that the first integral in (4) is for large enough values of $n$ less in absolute value than the sum of (11) and (12). Hence it approaches zero as $n$ becomes infinite, and thus our lemma is proved for this case. As stated before, the proof for the other case is analogous.

LEMMA 2. *The function $J_\nu(\alpha x)$, where $\alpha$ is a real constant $\neq 0$, can be expanded into a series of the form*

$$\sum_{n=1}^{\infty} B_n F_\nu(\lambda_n, x), \tag{13}$$

* The ambiguous signs enclosed in parentheses form an independent set from those not so enclosed.

† For the corresponding values of $z$ the integrand of the second integral in (4) can be shown to be less in absolute value than

$$\frac{12}{\sqrt{x}(n\pi+\gamma)^2}.$$

*where* $\lambda_1, \lambda_2, \lambda_3, \cdots$ *are the roots, positive or zero, of equation* (3) *for the case* $l \neq 0$, *arranged in increasing order of magnitude, the B's are constants, and the series converges uniformly in the interval* $0 \leqq x \leqq 1$.

Let us set

$$\frac{\omega(z)}{z} = u(z) = \frac{v(z)}{z^{\nu+1}}. \tag{14}$$

Then if we make use of Cauchy's theorem of residues, we obtain for the first integral in (4)

$$\frac{1}{2\pi i}\int_{R_n} \frac{F_\nu(z, x)}{(z-\alpha)u(z)}\,dz = \frac{F_\nu(\alpha, x)}{u(\alpha)} + \sum_{\lambda} \frac{F_\nu(\lambda, x)}{(\lambda-\alpha)u'(\lambda)}, \tag{15}$$

where the summation on the right-hand side is extended over all values of $\lambda$ that are roots of the first order of $u(\lambda)$ and lie within the rectangle $R_n$.

Since for every positive root of (3) there is a corresponding negative root, equal in numerical value, there will be two terms on the right-hand side of (15) corresponding to each positive root of (3). We shall now combine these two terms into one of the same type as the general term of (13).

Since from (14) any positive root of $\omega(z)$ is also a positive root of $u(z)$ and $v(z)$, we have for any positive root $\lambda$ of (3)

$$u'(\lambda) = \frac{v'(\lambda)}{\lambda^{\nu+1}} - \frac{(\nu+1)v(\lambda)}{\lambda^{\nu+2}} = \frac{v'(\lambda)}{\lambda^{\nu+1}}. \tag{16}$$

But from (14) and (9)

$$v'(\lambda) = (l\nu + h)J_\nu'(\lambda) - lJ_{\nu+1}(\lambda) - l\lambda J'_{\nu+1}(\lambda),$$

which by means of (8) and another well-known formula,

$$J'_{\nu+1}(z) = J_\nu(z) - \frac{\nu+1}{z}J_{\nu+1}(z),$$

may be put in the form

$$v'(\lambda) = -hJ_{\nu+1}(\lambda) + \frac{l(\nu^2 - \lambda^2) + h\nu}{\lambda}J_\nu(\lambda). \tag{17}$$

Moreover, since $\lambda$ is a positive root of (3) and the $l$ of (3) has been assumed to be not zero, it follows from (9) that

$$J_{\nu+1}(\lambda) = \frac{l\nu + h}{l\lambda}J_\nu(\lambda). \tag{18}$$

Combining (16), (17) and (18), we obtain

$$u'(\lambda) = \frac{l^2(\nu^2 - \lambda^2) - h^2}{l\lambda^{\nu+2}} J_\nu(\lambda). \tag{19}$$

It follows from (1) and (19) that

$$F_\nu(-\lambda, x) = F_\nu(\lambda, x), \qquad u'(-\lambda) = u'(\lambda).$$

Hence the two terms on the right-hand side of (15) corresponding to $\lambda$ and $-\lambda$ combine into the form

$$\frac{2\alpha l\lambda^2}{(\lambda^2 - \alpha^2)\{l^2(\nu^2 - \lambda^2) - h^2\}F_\nu(\lambda, 1)} F_\nu(\lambda, x). \tag{20}$$

We know from Lemma 1 that for values of $x$ in the interval $0 < x < 1$ the left-hand side of equation (15) approaches zero as $n$ becomes infinite. Hence the right-hand side does also, and if we take limits we obtain for $J_\nu(\alpha x)$ a series of the form (13),* where

$$B_n = \frac{2\alpha^{\nu+1}u(\alpha)l\lambda^2}{(\lambda^2 - \alpha^2)[l^2(\lambda^2 - \nu^2) + h^2]F_\nu(\lambda, 1)}, \tag{21}$$

and we know that the series converges to $J_\nu(\alpha x)$ in the interval $0 < x < 1$. It follows at once from the form of $B_n$ and well-known properties of the roots of (3) that the series converges *uniformly* throughout the interval $0 \leqq x \leqq 1$, and hence it represents $J_\nu(\alpha x)$ throughout that interval. Our lemma is therefore proved.

LEMMA 3. *The function $J_\nu(\alpha x) - x^\nu J_\nu(\alpha)$, where $\alpha$ is a real constant $\neq 0$, can be expanded into a series of the form* (13) *where $\lambda_1, \lambda_2, \lambda_3, \cdots$ are the successive positive roots of equation* (3) *for the case $l = 0$, arranged in increasing order of magnitude, the B's are constants, and the series converges uniformly in the interval $0 \leqq x \leqq 1$.*

If we treat the second integral in (4) in the same way as we treated the first integral in the proof of Lemma 2, we obtain for the function $J_\nu(\alpha x) - x^\nu J_\nu(\alpha)$ a series of the form (13) where

* It is apparent from (9) and (14) that $u(z)$ will have a zero root of the first order when and only when (3) has a zero root. As would be expected, the coefficient of the general term of the series (2) for the case where $f(x) = J_\nu(\alpha x)$ may be reduced to the form (21).

$$B_n = \frac{2\alpha^{\nu+2}F_\nu(\alpha, 1)}{\lambda^2(\lambda^2 - \alpha^2)F_{\nu+1}(\lambda, 1)}, \tag{22}$$

and we know that the series converges to the function in question throughout the interval $0 < x < 1$. It follows directly from the form of $B_n$ and well-known properties of the roots of $J_\nu(\lambda)$ that the series converges uniformly throughout the interval $0 \leqq x \leqq 1$. Hence it represents $J_\nu(\alpha x) - x^\nu J_\nu(\alpha)$ in that interval, and our lemma is proved.

LEMMA 4. *If in the interval* $0 \leqq x \leqq 1$, $\psi(x)$ *is defined and such that* $x\psi(x)$ *has a Lebesgue integral in this interval, and if furthermore*

$$\int_0^1 x\psi(x)F_\nu(\lambda_n, x)dx = 0 \quad (n = 1, 2, 3, \cdots), \tag{23}$$

*where* $\lambda_1, \lambda_2, \lambda_3, \cdots$ *are the successive roots, positive or zero, of equation* (3), *then* $\psi(x)$ *is zero at every point of the interval* $0 < x < 1$ *at which it is continuous; it is zero for* $x = 1$ *if continuous for that value and the* $l$ *of equation* (3) $\neq 0$, *or if it is continuous at that point and at an infinite number of points in its neighborhood; it is zero for* $x = 0$ *if it is continuous at that point and at an infinite number of points in its neighborhood.*

The proof is somewhat different in the two cases where the $l$ of equation (3) is different from or is equal to zero. We shall consider first the former case.

By Lemma 2 the function $J_\nu(\alpha x)$ may be expanded into a series of the form (13) that converges uniformly in the interval $0 \leqq x \leqq 1$. Multiplying the series by $x\psi(x)$ and integrating term by term* from 0 to 1, we obtain in view of (23)

$$\int_0^1 x\psi(x)J_\nu(\alpha x)dx = 0. \tag{24}$$

If in this equation we replace $J_\nu(\alpha x)$ by its development in a power series, integrate term by term,† and divide through by $\alpha^\nu$, we obtain

$$0 = \sum_{i=0}^{\infty} \frac{(-1)^i\alpha^{2i}}{2^{\nu+2i}i!\,\Gamma(\nu + i + 1)} \int_0^1 x^{\nu+1+2i}\psi(x)dx,$$

* It follows from the uniform convergence of the series for $J_\nu(\alpha x)$ and the fact that $x\psi(x)$ has a Lebesgue integral in the interval $0 \leqq x \leqq 1$ that we may obtain the value of the integral on the left hand side of (24) by integrating term by term.

† The justification is the same as before.

and this equation is known to hold for all real values of $\alpha$ except zero. Since both sides of the equation are analytic functions, it follows that the equation holds for all values of $\alpha$ and therefore that the right-hand side is identically zero. Hence we have

$$\int_0^1 x^{\nu+1+2i}\psi(x)dx = 0 \quad (i = 1, 2, 3, \cdots), \tag{25}$$

and therefore $\psi(x)$ is zero at every point of the interval $0 < x \leqq 1$ at which it is continuous,* and is zero for $x = 0$ if continuous at that point and at an infinite number of points in its neighborhood.

Let us now consider the case where the $l$ of equation (3) is zero. Making use of Lemma 3 in the same way as we made use of Lemma 2 in the discussion of the previous case, we obtain in place of equation (25) the relation

$$\int_0^1 x^{\nu+1}(x^{2i} - 1)\psi(x)dx = 0 \quad (i = 1, 2, 3, \cdots).$$

From this we readily obtain

$$\int_0^1 x^{\nu+1+2i}(x^2 - 1)\psi(x)dx = 0 \quad (i = 0, 1, 2, \cdots).$$

Hence $\psi(x)$ is zero at every point of the interval $0 < x < 1$ at which it is continuous, and is zero for $x = 0$ or $x = 1$ if continuous at these points and at an infinite number of points in their respective neighborhoods. Thus our lemma is completely proved.

We are now ready to prove the following theorem.

THEOREM: *If the function $f(x)$ defined for the interval $0 \leqq x \leqq 1$ is such that $xf(x)$ has a Lebesgue integral in this interval, and if the series of the form* (2) *corresponding to $f(x)$ converges or is summable at all points of the interval $0 \leqq x \leqq 1$ at which $f(x)$ is continuous, defines a function $\varphi(x)$ that is continuous at those points† and such that $x\varphi(x)$ has a Lebesgue*

* Cf. Lerch, *Acta Mathematica*, vol. 27 (1903), p. 347. Also the writer, BULLETIN, vol. 14 (1908), p. 368 and vol. 15 (1908), p. 116. The proof of the theorem referred to is given in these papers for Riemann integrability; it applies equally well for Lebesgue integrability except at the point $x = 0$, where our added condition is necessary to complete the proof in the same way as it is completed in the writer's footnote on p. 371 of the first reference above.

† The point $x = 1$ is to be excepted for the case $l = 0$, unless $f(1) = 0$.

*integral in the same interval, and can be integrated term by term over this interval after being multiplied through by a function that is continuous there, then the series will converge or be summable to $f(x)$ at all points of the interval $0 < x < 1$ at which $f(x)$ is continuous;* it will converge or be summable to $f(x)$ for $x = 1$ if $f(x)$ is continuous there and $l \neq 0$, or if $f(x)$ is zero for $x = 1$, and is continuous at that point and at an infinite number of points in its neighborhood; it will converge or be summable to $f(x)$ for $x = 0$ if the function is continuous at that point and at an infinite number of points in its neighborhood.*

We have

$$\varphi(x) = \sum_{n=1}^{\infty} F_\nu(\lambda_n, x) \frac{\int_0^1 xf(x)F_\nu(\lambda_n, x)dx}{\int_0^1 x[F_\nu(\lambda_n, x)]^2 dx}.$$

Multiplying by $xF_\nu(\lambda_n, x)$ for $n = 1, 2, 3, \cdots$ and integrating from 0 to 1, we obtain, in view of the fact that the $F$'s form an orthogonal set,

$$\int_0^1 x[\varphi(x) - f(x)]F_\nu(\lambda_n, x)dx = 0 \quad (n = 1, 2, 3, \cdots).$$

From this equation and Lemma 4 our theorem follows at once.

UNIVERSITY OF CINCINNATI,
*June*, 1916.

---

## SECOND NOTE ON REMOVABLE SINGULARITIES.

BY DR. W. E. MILNE.

IN a "Note on removable singularities"† the writer stated a theorem‡ concerning removable singularities for functions of several complex variables. An analogous theorem, with less restrictive hypotheses, is the following

* The convergence or summability to $\frac{1}{2}[f(x+0) + f(x-0)]$ at points where $f(x)$ has a finite jump does not follow directly from the present method. It can be obtained from the convergence or summability to $f(x)$ at points of continuity by a method given by the writer on pp. 428–429 of the paper in volume 10 of the *Transactions*, referred to above.

† BULLETIN, vol. 21 (1914), pp. 116–117.

‡ The proof there given for this theorem is incomplete, as Dr. Dunham Jackson pointed out. The gap is filled by the proof here given.

THEOREM. *Let the function* $\varphi(z_1, \cdots, z_n)$ *of* $n$ *complex variables,* $n > 2$, *be analytic in the region* $(S) = (S_1, \cdots, S_n)$ *except for the points of a manifold* $M$ *of such character that, when the* $n - 2$ *variables* $z_3, \cdots, z_n$ *are given any fixed values in their respective regions, the points of* $M$ *whose last* $n - 2$ *coordinates are the above* $z_3, \cdots, z_n$ *are isolated, and thus their first two coordinates* $z_1, z_2$ *lead to isolated points in* $(S_1, S_2)$. *Then the function* $\varphi$ *has a limit in the points of* $M$, *and if defined as equal to its limit will be analytic without exception in* $(S)$.

*Proof.* Let $P : (a_1, \cdots, a_n)$ be any point on $M$, and hold $z_3, \cdots, z_n$ fast at $(a_3, \cdots, a_n)$. Since the points in $(S_1, S_2)$ corresponding to $(a_3, \cdots, a_n)$ (i. e., points $(z_1, z_2)$ such that the points $(z_1, z_2, a_3, \cdots, a_n)$ are points of $M$) are isolated, it is possible to draw circles $c_1$ about $a_1$ and $c_2$ about $a_2$ small enough to exclude all the points thus arising except $(a_1, a_2)$. Let $t_1$ be on $c_1$ and $t_2$ on $c_2$. Then $\varphi$ will be analytic at $(t_1, t_2, a_3, \cdots, a_n)$ since this point is not on $M$, and hence to each $(t_1, t_2)$ corresponds a positive number $K$ such that $\varphi(t_1, t_2, z_3, \cdots, z_n)$ is analytic in $z_3, \cdots, z_n$ when $|z_i - a_i| < K$. Since the set of points $(t_1, t_2)$ is closed it is easily seen that the lower limit $K_0$ of the corresponding set of $K$'s is not zero, so that $\varphi(t_1, t_2, z_3, \cdots, z_n)$ is analytic in $z_3, \cdots, z_n$ when $|z_i - a_i| < K_0$, $i = 3, \cdots, n$, for *every* point $(t_1, t_2)$ on $c_1$ and $c_2$.

Now $\varphi$ may be so defined on $M$ as to be analytic in $z_1$ and $z_2$ throughout $(S_1, S_2)$ for every choice of $z_3, \cdots, z_n$ in $(S_3, \cdots, S_n)$, because the singularities in $(S_1, S_2)$ are isolated, and isolated singularities for a function of two complex variables are removable. Therefore if $z_1$ is inside $c_1$ and $z_2$ inside $c_2$ we have

$$\varphi(z_1, \cdots, z_n) = \left(\frac{1}{2\pi i}\right)^2 \int_{c_1}\int_{c_2} \frac{\varphi(t_1, t_2, z_3, \cdots, z_n)dt_1dt_2}{(t_1 - z_1)(t_2 - z_2)}.$$

The right-hand side of this equation is analytic in all $n$ variables together at the point $P$, and therefore $\varphi$ is analytic in all $n$ variables at $P$. Since $P$ was any point on $M$, the theorem is established.

*Remark.* Two important classes of singular manifolds to which the above theorem will apply were suggested to me by Professor Osgood.

The first is the class of manifolds defined in the neighborhood of a point by two equations

$$F_1(z_1, \cdots, z_n) = 0, \qquad F_2(z_1, \cdots, z_n) = 0,$$

where $F_1$ and $F_2$ are each analytic at the point in question and vanish there, but have no common factor there. Suppose for simplicity that the point is the origin. Then by Weierstrass's theorem of factorization it will be possible, at least after a non-singular linear transformation of the independent variables, to write

$$F_1 = [z_1^m + A_1 z_1^{m-1} + \cdots + A_m]\theta(z_1, \cdots, z_n),$$

$$F_2 = [z_1^\mu + B_1 z_1^{\mu-1} + \cdots + B_\mu]\psi(z_1, \cdots, z_n),$$

where the $A$'s and the $B$'s are functions of $z_2, \cdots, z_n$ which are analytic at the origin and vanish there, while $\theta$ and $\psi$ are analytic at the origin and do not vanish there. The resultant of the polynomials in brackets is a function $R(z_2, \cdots, z_n)$ analytic at the origin and vanishing there, but not identically zero. Hence we can make a linear transformation of $z_2, \cdots, z_n$ so that $R(z_2, 0, \cdots, 0)$ is not identically zero. Now it will be seen that when $z_3, \cdots, z_n$ are all held fast at the origin the point $z_1 = 0, z_2 = 0, \cdots, z_n = 0$, is an isolated point of the manifold $M$. For in a sufficiently small neighborhood of the origin in the $z_2$-plane the function $R(z_2, 0, \cdots, 0)$ vanishes only when $z_2 = 0$; and when $z_2 = 0, z_3 = 0, \cdots, z_n = 0$, the only common solution of $F_1$ and $F_2$ in the neighborhood of the origin in the $z_1$-plane is $z_1 = 0$.

The second class is the class of $(2n - 4)$-dimensional real manifolds which are *regular* at the point in question (which we may again take as the origin). For then $M$ is defined by the equations

$$z_i = \varphi_i(t_1, \cdots, t_m) + \sqrt{-1}\psi_i(t_1, \cdots, t_m),$$
$$(i = 1, 2, \cdots, n; \quad m = 2n - 4),$$

where the $\varphi$'s and $\psi$'s are single-valued real functions of the real parameters $t_1, \cdots, t_m$ which vanish at the origin and have continuous first partial derivatives there. Moreover the matrix

$$a = \left\| \begin{matrix} \frac{\partial \varphi_1}{\partial t_1} & \cdots & \frac{\partial \varphi_1}{\partial t_m} \\ \cdot & \cdot & \cdot \\ \frac{\partial \varphi_n}{\partial t_1} & \cdots & \frac{\partial \varphi_n}{\partial t_m} \\ \frac{\partial \psi_1}{\partial t_1} & \cdots & \frac{\partial \psi_1}{\partial t_m} \\ \cdot & \cdot & \cdot \\ \frac{\partial \psi_n}{\partial t_1} & \cdots & \frac{\partial \psi_n}{\partial t_m} \end{matrix} \right\|$$

is of rank $m$ at the origin. A general homogeneous linear transformation of the $z$'s,

$$T: \qquad z_i' = \sum_{j=1}^{n} a_{ij} z_j,$$

yields a linear transformation $T'$ of the $\varphi$'s and $\psi$'s of the form

$$T': \qquad \varphi_i' = \sum_{j=1}^{n} (\alpha_{ij}\varphi_j - \beta_{ij}\psi_j),$$

$$\psi_i' = \sum_{j=1}^{n} (\beta_{ij}\varphi_j + \alpha_{ij}\psi_j),$$

where $a_{ij} = \alpha_{ij} + \sqrt{-1}\beta_{ij}$. If $T'$ is non-singular then $T$ is non-singular. Let $\delta$ be the matrix of the transformation $T'$ and $\bar{a}$ the matrix corresponding to $a$ after the transformation. Then

$$\bar{a} = \delta \times a.$$

The $m$-rowed determinants in $\bar{a}$ are polynomials in the $\alpha_{ij}$ and $\beta_{ij}$ which are not identically zero, since $a$ is of rank $m$. Therefore a particular transformation $T'$ may be chosen so that all the $m$-rowed determinants in $\bar{a}$ are different from zero and also so that $T'$ is non-singular when $t_1 = t_2 = \cdots = t_m = 0$. Then after the transformation $T$ corresponding to $T'$ has been made, if $z_3, \cdots, z_n$ are all held fast at the origin, the point $z_1 = 0, z_2 = 0$ is an isolated point of $M$.

BOWDOIN COLLEGE,
*April*, 1916.

# THE MITTAG-LEFFLER TESTAMENT AND INSTITUTE.*

WE the undersigned, modifying our joint testament of January 6, 1883, do here declare our last will, which is to bequeath all our possessions, after our deaths, to a foundation bearing the name Mittag-Leffler Mathematical Institute.

This institute shall have for its purpose to maintain and develop still further the position now held by pure mathematics in the four Scandinavian countries of Sweden, Denmark, Finland, and Norway, but especially in Sweden, and also to make known and appreciated at its true value beyond their borders the share of these countries in the highest sphere of intellectual life.

In the accomplishment of this task, no objects other than those mentioned above shall be considered. No account shall be taken of personal friendships, or of the desire to lend pecuniary assistance to those in need. Neither shall any consideration be given to the needs of practical life, to examination questions, to political opinions, or to the aims of any science other than *pure* mathematics.

The Institute shall accomplish its task as follows:

1. By preserving and enriching the mathematical library of the undersigned G. Mittag-Leffler, with all that appertains to it in the way of manuscripts, portraits, family collections and souvenirs, and other objects.

The library shall remain in the large stone villa situated on our property in quarter No. 16, called Mitgård, at Djursholm, and shall not be incorporated in any other collection of books. The villa has been built and arranged with this purpose in view, and contains several rooms in which readers can work without disturbance.

The small part of the villa which is now used as a dwelling shall also be devoted, after our deaths, to the library.

The library shall be open to all mathematicians, but, to prevent abuses, only under the authorization of the president of the executive committee of the Institute. The books

* The following extract from the Mittag-Leffler testament was published on the occasion of the celebration at Stockholm of Dr. G. Mittag-Leffler's seventieth birthday, March 16, 1916. It is translated for the BULLETIN by Dr. CAROLINE E. SEELY.

shall not be lent out, and shall be used only in the library building.

2. By granting fellowships, for study at home or abroad, to young people of both sexes, belonging to the four countries above named, who have given evidence of real aptitude for research and discovery in the domain of *pure* mathematics.

Moreover, works of unusual importance, written by inhabitants of these countries, may be made the objects of special recognition, in the form of a gold medal, of the same form and design as the small Nobel medal, and as long as copies shall remain, of a set of the *Acta Mathematica*, in a special binding bearing the name of the author crowned.

3. By awarding prizes for discoveries really worthy of the name in the domain of *pure* mathematics. These prizes shall be given without regard to the nationality of the author. He may belong to any country whatever, and inhabitants of the four Scandinavian countries shall not enjoy, in this matter, any privilege. The prize shall be awarded only for a discovery bringing with it new ideas of such a nature that they give a new impetus to the science. It is desirable, however, that the prize be awarded once at least in every six years. This prize shall consist of a large gold medal and a diploma, of artistic design, the latter stating the scientific reasons for the award, and of as complete a set as possible of the *Acta Mathematica*, in special binding bearing the name of the recipient. He shall be invited to appear personally at Djursholm to receive the prize, a suitable appropriation for his travelling expenses being made. The prize shall be presented at a special ceremony in the great hall of the library.

4. When the annual revenues of the Institute shall exceed the amount indicated below, there may be created, beside the post of director, other remunerated positions, whose holders shall devote themselves to exclusively scientific writing and teaching in the domain of pure mathematics.

To the preceding provisions the following shall be added:

A. The Executive Committee of the Institute shall be composed of the Swedish members of the first class (pure mathematics) of the Royal Academy of Sciences, together with Professors Ivar Fredholm and N. E. Nörlund, during their lifetimes. The Director of the Institute shall also be a member. The Committee may also add to its membership, for a suitable period, any Swedish mathematician of real

eminence who shall agree entirely with the motives that have actuated us but who shall not yet have become a member of the first class of the Academy of Sciences. There may also be added to the Committee a mathematician from one of the three other Scandinavian countries, under the same conditions.

B. As soon as possible, there shall be appointed, to fill the post of scientific and executive Director of the Institute, a mathematician of eminent rank, who shall appear particularly qualified for this charge, and whose activities shall be devoted exclusively to personal research and shall contribute, at the same time, to the ends pursued by the Institute. He shall also, when suitable occasion arises and for strictly scientific purposes, give courses of lectures for a limited number of really gifted auditors.

From the material standpoint, he shall be placed in a more advantageous position than any professor of mathematics in any University in the four Scandinavian countries. He shall be domiciled at Djursholm, and if possible in the immediate neighborhood of the library. An appropriation shall be made for his house rent until a special building shall have been fitted up for his use. His nomination shall be made, on recommendation of the Committee, by His Majesty the King, if, as we venture to hope, His Majesty deigns to consent.

. . . . . .

E. At least once in every six years, the Institute shall hold a special meeting. The mathematicians of the four Scandinavian countries shall receive a personal invitation to attend. We hope that, in consideration of the importance of the Institute for these countries, all will be present if possible.

It is to be desired that the day of this meeting be chosen to coincide with the date of the meeting of the congress of Scandinavian mathematicians at Stockholm. On this occasion shall be read a report of the activities of the Institute since its last special meeting. The ceremony shall be invested with a character of dignity that shall emphasize plainly the elevated mission of the mathematical sciences, as well as the purpose of the activities of the Institute.

Finally, I, the undersigned G. Mittag-Leffler, wish to declare that the model which I have had in mind for the Institute founded by my wife and myself, is the Institut Pasteur at Paris. It seems to me that this Institute has fulfilled the

mission of a center of scientific research better than any university or academy. The universities have everywhere, in addition to their scientific task, another that often interferes very seriously with it, namely the training of teachers and officials. As for the academies, which have been the most effective institutions from the scientific standpoint up to the present time, they suffer from two disadvantages: on the one hand, the main activities of most of their members are not confined to their interests, and on the other hand, when this is not the case, the scientist lacks the stimulus to his own investigations to be found in guiding or assisting the researches of others. Our Institute is not connected with an establishment where experimental research can be carried on, but on the contrary, in conformity with the needs of pure mathematics, with a special and very complete library.

One may find in our country sufficient possibilities for the creation and organization of institutes for the natural sciences, but very few people, apart from specialists, understand the importance and the mission of pure mathematics. That is why the undersigned, G. Mittag-Leffler, has always desired to be able to found an institute like that which we hope to have established by this testament.

Our testament owes its origin to the lively conviction that a people that does not accord to mathematics a high place in its estimation, will never be in a position to fulfil the most lofty tasks of civilization, and to enjoy, in consequence, that international consideration which is itself, in the end, an effective means of preserving our place in the world and of safeguarding our right to live our own life.

The testament contains finally some directions to the effect that the Institute shall enter upon its activities on the death of G. Mittag-Leffler, except for certain life interests to be assured to Madame Signe Mittag-Leffler, and certain directions with regard to the administration of the estate and to certain small life interests and other allowances.

# THE MADISON COLLOQUIUM LECTURES ON MATHEMATICS

*Part II. Topics in the Theory of Functions of Several Complex Variables.* By WILLIAM FOGG OSGOOD. New York, American Mathematical Society, 1914.

THE transition from the theory of functions of a single complex variable to the analogous theory for two variables corresponds in the domain of reals to a transition from functions on a plane to functions whose arguments are points of a four dimensional space. Generalizations from two to three real dimensions frequently involve great difficulties, even with the aid of our geometrical intuitions of the plane and three-space. It is not surprising therefore to find that familiar theorems concerning functions of a single complex variable either have no analogues at all, or else yield a multiplicity of generalizations, when the number of independent variables is increased. In the plane, for example, we have only the theory of connectivity associated with curves, while in space of four dimensions there are three kinds of connectivity corresponding in an analogous way to curves, surfaces, and three-spreads. On these subspaces of different dimensions there are theories of integrals corresponding to the curve integrals which play such a fundamental rôle in the theory of functions of a single complex variable.

The increase in complexity mentioned in the preceding paragraph arises from the greater multiplicity of geometrical configurations in the higher spaces. But there are also serious analytical obstacles to the extensions of the theory of functions of a single complex variable. Thus for a function $f(z_1, z_2)$ of two variables $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ the familiar potential equation is replaced by the system

$$\frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {y_1}^2} = 0, \quad \frac{\partial^2 u}{\partial {x_2}^2} + \frac{\partial^2 u}{\partial {y_2}^2} = 0,$$

$$\frac{\partial^2 u}{\partial x_1 \partial y_2} - \frac{\partial^2 u}{\partial x_2 \partial y_1} = 0, \quad \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial y_1 \partial y_2} = 0,$$

which admits of a much greater variety of initial conditions and is in other respects also more formidable.

A typical example of a theorem which breaks down entirely in the higher spaces is the following: Let $\varphi(x, y)$ be a function of two complex variables analytic at the origin and having $\varphi(0, y) \equiv 0$. With the help of the preparation theorem of Weierstrass $\varphi$ can always be decomposed into a product of the form

$$\varphi(x, y) = P(x, y)M(x, y),$$

where $P(x, y)$ is a polynomial in $y$ with coefficients which are series in $x$ without constant terms, while $M$ is a series in $x$ and $y$ with non-vanishing constant term. In a suitably chosen neighborhood of the origin, therefore, the function $y(x)$ defined by the equation $\varphi = 0$ is finitely multiple valued and has its values defined by the polynomial equation $P(x, y) = 0$. In the Colloquium Lectures Professor Osgood raises the question as to whether a similar theorem would be true for a function $y(x_1, x_2, \cdots, x_n)$ defined by an equation of the form

$$\varphi(x_1, x_2, \cdots, x_n, y) = 0. \tag{1}$$

He has more recently discovered that the theorem is not true when $n > 1$,* and the example by means of which he proves this statement provides some very interesting further information. An equation of the form (1) may in fact define a function $y(x_1, \cdots, x_n)$ which is single-valued and monogenic in a part of a neighborhood of the origin, while the origin itself is a point of a natural boundary of the function, other parts of the neighborhood being lacunary spaces. Such a function could clearly not be analytic or even algebroid at the origin.

The foregoing remarks will indicate to some extent the complexity of the questions which lie at the foundations of the theory of functions of several complex variables. Professor Osgood has been a thorough student of the subject with a keen appreciation of its deficiencies at the present stage of development. His Lectures seem to be a resumé preliminary to the publication of an exhaustive treatise, and it is much to be hoped that such a treatise may appear from his pen in the near future. Lecture I contains a historical introduction and "general survey" of the field, while the other four lectures are devoted to a descriptive and critical exposition of fundamental theorems, some of the most sig-

* *Transactions Amer. Math. Society*, vol. 17 (1916), p. 4.

nificant of which are the result of the author's own researches. The whole will be most helpful to students of the subject.

In the first lecture the author mentions a definition of a function of several complex variables published by Cauchy in 1831, but the earliest phases of the theory to be studied in detail were associated with special problems. Among the more important of these were the extensions of the existence theorems of Cauchy to partial differential equations and systems of equations defining implicit functions of several complex variables, the periodic abelian functions arising from Jacobi's problem of inversion, and periodic functions of several variables in general. The existence theorems did not require the development of new methods, their extensions being obtained by means of the famous device of dominating series applied by Cauchy to the simpler cases. The masterly generalization of the elliptic inversion problem by Jacobi was, however, the precursor of a long series of investigations during one of the most fruitful periods in the annals of mathematics. It was during this period that the theory of the theta functions of several variables, which lies at the heart of the periodic function theory, was so highly developed by Riemann and Weierstrass.

The results which have just been mentioned were the products of successful attempts to generalize the elliptic functions and elliptic theta functions. After the development of the theory of automorphic functions of a single variable by Klein and Poincaré in the early eighties, it was natural that similar generalizations should be sought for functions of several variables. Picard was the pioneer in this field. The functions studied when, for example, there are two independent complex variables $x$, $y$ are those which are invariant under groups of projective transformations, or groups of transformations of the form

$$x' = \frac{\alpha_1 x + \beta_1}{\gamma_1 x + \delta_1}, \quad y' = \frac{\alpha_2 y + \beta_2}{\gamma_2 y + \delta_2}.$$

The theory of these functions in turn stimulated the interest of Picard and his associates in algebraic functions of two variables and the surface and line integrals of various kinds associated with them. Their results are analogous in a general way to those of the theory of algebraic functions of a single variable, although the theorems in particular cases are strikingly different.

Lecture II is devoted to a rather heterogeneous collection of theorems. It begins with an abbreviated account of a generalization of Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t - z}$$

and the theory of residues. Several sections are then devoted to a discussion of theorems giving sufficient conditions that a function of several complex variables be analytic, rational, or algebraic. Weierstrass stated the theorem that a function $f(z_1, z_2, \cdots, z_n)$ of several complex variables is surely rational if it is representable near every point of analysis by the quotient of two functions analytic and prime to each other at the point in question. He did not specify explicitly the character of the infinite region implied in the phrase "at every point of analysis," but he seems to have intended that each of the variables $z_1, z_2, \cdots, z_n$ should range independently over a plane extended by a single point at infinity. Professor Osgood has called attention to the fact that the part of the hypothesis of the theorem referring to the infinite region is under these circumstances equivalent to presupposing properties of the function $f$ at finite points into which the points of the infinite region have been converted by a transformation of the form

$$z_k' = \frac{\alpha_k z_k + \beta_k}{\gamma_k z_k + \delta_k} \qquad (k = 1, 2, \cdots, n).$$

It is a theorem due to himself that a conclusion similar to that of Weierstrass can be drawn if the transformation just written is replaced by a projective transformation on the $n$ variables $z$. This involves a conception of the points at infinity different from that of Weierstrass, and there are gradations between the two, with possibly similar theorems, if the variables are transformed one part projectively and the rest bilinearly. In Section 6 of this lecture Professor Osgood draws the very interesting conclusion that similar results can be obtained by assuming properties of the function $f$ at a suitably characterized portion only of the region at infinity.

Let a power series in $x, y$ be convergent for $|x| < r$, $|y| < s$, and let $s$ be the largest positive constant for which, when $r$ is fixed, the series has this convergence property. Then $s$ is

a function $s = \varphi(r)$ called the radius of convergence associated with $r$. Sections 7 and 8 of this lecture contain a summary of the known properties of the function $\varphi(r)$ and theorems related to it.

The last part of Lecture II describes generalizations of the famous theorems of Weierstrass and Mittag-Leffler concerning the construction of a function of a single complex variable having assigned poles and zeros in a given continuum. There always exists such a function analytic except at the poles, expressible as a quotient of two other functions each analytic at every point of the continuum. The essence of the generalization to functions of several complex variables lies in the characterization of the points which correspond to poles or zeros. Expressed very roughly the theorem is that if at every point $a = (a_1, a_2, \cdots, a_n)$ of the space a quotient

$$f_{(a)}(z_1, z_2, \cdots, z_n) = \frac{H_{(a)}(z_1, z_2, \cdots, z_n)}{G_{(a)}(z_1, z_2, \cdots, z_n)} \tag{2}$$

is assigned, where $H$ and $G$ are analytic and prime to each other at the point in question, then there exists a quotient $f = H/G$ of two permanently convergent power series which is equivalent at every point $a$ to the corresponding quotient $f_{(a)}$. Two functions are equivalent at a point $a$ if their quotients, taken both ways, have only removable singularities in a neighborhood $T_{(a)}$ of the point. Professor Osgood describes also a more general theorem of Cousin, and some consequences involving generalizations by Gronwall of the Weierstrassian notion of prime factors of integral functions.

The third lecture is concerned with singular points and analytic continuation. A non-essential singularity of the first kind, sometimes called a pole, is a point $(a_1, a_2, \cdots, a_n)$ at which the function is expressible in the form (2) with $H_{(a)} \neq 0$, $G_{(a)} = 0$ at $(a_1, a_2, \cdots, a_n)$; the point is a non-essential singularity of the second kind if $H_{(a)}$ and $G_{(a)}$ both vanish but have no vanishing factors in common there; all other non-removable singularities are essential.

Professor Osgood gives a number of theorems, by Kistler and Hartogs, specifying conditions under which a singularity can be removed after assigning suitably chosen values to the function at the point itself and at other exceptional points near it. No non-removable singularities are isolated when $n > 1$, and in particular every essential singularity is a limit

point of others of the same type. Starting from a theorem concerning analytic continuation Professor Osgood shows how Hartogs has proved that if a function $f(x, y)$ of two complex variables is analytic except at the points of a two-dimensional continuous locus $y = \varphi(x)$ consisting of non-removable singularities, then $\varphi(x)$ itself must be analytic. This is a sharp contrast to the fact that the singularities of a function of a single complex variable may have a distribution as arbitrary as the boundary of an arbitrarily selected two-dimensional continuum.

Another striking theorem, deducible from a theorem of Levi which Professor Osgood characterizes as one of the most important contributions to the theory in recent years, is that a function $f(x, y)$ analytic (or meromorphic) at all points of the boundary of a finite and regular four-dimensional region $T$, can always be continued analytically (or meromorphically) throughout $T$. This has as a consequence that such a function can not have a finite lacunary region around which it is analytic. A three-dimensional locus

$$\varphi(x_1, x_2, y_1, y_2) = 0,$$

where $x = x_1 + ix_2$, $y = y_1 + iy_2$, can not be arbitrarily selected if it is to be a natural boundary for a function $f(x, y)$; it must satisfy a certain differential inequality. Levi-Civita has shown further that a function of two complex variables is in general uniquely determined if its values are assigned along a two-dimensional locus. But there are exceptional "characteristic" loci along which arbitrarily assigned values may determine none or an infinity of functions, a situation which reminds us forcibly that we are dealing more or less directly with initial values for solutions of a system of partial differential equations.

In Lecture IV Professor Osgood discusses certain questions in implicit function theory based upon the preparation theorem of Weierstrass. This theorem states that a function $F(u; x_1, x_2, \cdots, x_n)$ analytic at the origin and having a lowest term in the series $F(u; 0, 0, \cdots, 0)$ of order $m$, is expressible as a product

$$F = P\Omega, \tag{3}$$

where $\Omega$ is a series in $u, x_1, \cdots, x_n$ with constant term different from zero, while $P$ is a polynomial in $u$ of degree $m$ with

leading coefficient unity, its other coefficients being series in $x_1, \cdots, x_n$ vanishing at the origin. Near the origin the roots of $F$ are clearly all roots of the polynomial $P$, and vice versa. The theorem mentioned in an earlier paragraph, which Professor Osgood stated tentatively in the Lectures, and later disproved by an example, would have implied a generalization of this including the case when $F(u; 0, 0, \cdots, 0)$ vanishes identically.

The preparation theorem gives information with respect to the roots of a single equation $F = 0$ in a neighborhood of the origin, which is completely satisfactory for many purposes. For a system of equations

$$\Phi_k(u_1, \cdots, u_l; x_1, \cdots, x_p) = 0 \quad (k = 1, 2, \cdots, l), \tag{4}$$

where the functions $\Phi_k$ are also analytic and vanish at the origin, Weierstrass has given a further important theorem, described in Section 6 of Lecture IV. He was interested in characterizing the manifold of $(l + p)$-dimensional points, rather than in the implicit functions $u$ of the variables $x$ defined by equations (4).

The relation between $F$ and the polynomial $P$ in equation (3) could equally well have been expressed in the form $P = MF$, where $M = 1/\Omega$ is a series with constant term different from zero. In a paper published in 1912* the author of this review has shown that when the homogeneous polynomials of lowest degree, the "characteristic" polynomials, of the series $\Phi_k(u_1, \cdots, u_l; 0, \cdots, 0)$, have a resultant $R$ different from zero, there always exists a polynomial $P(u_l; x_1, \cdots, x_p)$ expressible linearly in the form

$$P = M_1\Phi_1 + M_2\Phi_2 + \cdots + M_l\Phi_l,$$

and such that every root system $(u_1, \cdots, u_l; x_1, \cdots, x_p)$ of equations (4) near the origin has values $(u_l; x_1, \cdots, x_p)$ making $P = 0$; while every sufficiently small root system of $P = 0$ is part of at least one solution of these equations near the origin in the $(l + p)$-dimensional space. The degree $N$ of $P$ in $u_l$ is the product of the degrees of the characteristic polynomials. Further when the substitution

* A generalization of Weierstrass's preparation theorem for a power series in several variables, *Transactions Amer. Math. Society*, vol. 13 (1912), p. 133.

$$(5) \qquad u_l = -t_1u_1 - t_2u_2 - \cdots - t_{l-1}u_{l-1} - z$$

is made in equations (4), the polynomial corresponding to $P$ has the form

$$Q(z;\ x_1, \cdots, x_p) = \prod_{i=1}^{N} (z - t_1u_{i1} - t_2u_{i2} - \cdots - t_{l-1}u_{i(l-1)} - u_{il}),$$

where the systems $(u_{i1}, \cdots, u_{il};\ x_1, \cdots, x_p)$ are the root systems of (4) corresponding to $x_1, \cdots, x_p$. The introduction of the variables $t$ is a well-known device of the algebraic elimination theory serving to unite the $u$-values which belong to a single root system of the equations.

It seems to me that the theorems described by Professor Osgood in Section 7 of Lecture IV could be deduced at once from the properties of the polynomial $Q$. For example, since $Q$ is of the $N$th degree, the system (4) must have exactly $N$ root systems near the $ux$-origin for each point $(x_1, \cdots, x_p)$ in a properly chosen neighborhood of the origin in the $x$-space. These systems will all be distinct unless $(x_1, \cdots, x_p)$ makes the $z$-discriminant of $Q$ identically zero in the variables $t$. The coefficients of the various $t$-terms are, however, series in $x_1, \cdots, x_p$ which set equal to zero define a locus analogous to Professor Osgood's "locus $D$," but not, if I understand him correctly, exactly the same. At every point of this locus some of the root systems of equations (4) coincide. Professor Osgood has stated his results for a more general hypothesis than mine. He requires only that the series $\Phi_k(u_1, \cdots, u_l;\ 0, \cdots, 0)$ shall have no common root near the origin except that point itself. The theorems of my paper apply, however, under very general circumstances, since the degrees of the characteristic polynomials are entirely arbitrary and their coefficients subjected to an inequality only. I must confess that my proofs are complicated, and I hope that this undesirable feature will be eliminated when Professor Osgood publishes in detail the theorems in his more general form.

On page 196 of the Lectures there is a reference to my paper which might lead one to infer that its character is similar to that of a paper by MacMillan. The hypotheses of the two papers are the same, but their principal theorems are entirely different. MacMillan discusses the highly interesting theorem, originally stated by Poincaré, that the solu-

tions of the equations (4) near the origin are the same as those of another system whose first members are polynomials in the variables $u$. The polynomials have in general extraneous roots which are not necessarily small. The polynomial $Q$ described above defining the solutions near the origin would be found from them by eliminating the $u$'s algebraically and applying the preparation theorem of Weierstrass to the resulting polynomial in $z$.

Lecture V describes a point of view in the algebraic function theory which seems most interesting and effective. Let $f(w, z) = 0$ be an irreducible algebraic equation of deficiency $p$, having a corresponding Riemann surface $F'$ rendered simply connected by period cuts. When $p = 1$ the equation $t = w(z)$, where $w$ is the integral of the first kind, maps $F'$ upon a curvilinear parallelogram in the $t$-plane, and the theory of functions and integrals on the Riemann surface then corresponds completely to that of a system of elliptic and allied functions with specified periods and $t$ as independent variable. The Weierstrassian function $\sigma(t)$ is the fundamental function for such a system, in terms of which all the others may be expressed. The theory is more uniform in the $t$-plane than on the Riemann surface, because the particular $t$-values which correspond to branch points of the surface are no more exceptional than other values of $t$.

For deficiencies $p > 1$ Klein has defined a "prime function" on the Riemann surface which plays a rôle analogous to that of the $\sigma$-function for the simpler case. Professor Osgood proposes to transfer the theory of this function to a $t$-plane on which the surface is mapped by a transformation $z = \varphi(t)$, the function $\varphi$ being a suitably selected automorphic function. He shows concisely but clearly in this lecture how the various types of functions analogous to those on the Riemann surface may be expressed in terms of the prime function $\Omega(t, \tau)$, and points out the advantages, in many cases decisive, to be gained by this uniformization.

It is not to be expected that the complications of a structure like that of the algebraic function theory can be entirely eliminated by a process such as Professor Osgood would have us use. But it is possible, on the other hand, that they may be so localized as not to mar the more facile beauty of the other parts of the theory. A first difficulty from his point of view seems to lie in the proof of the existence of the uni-

formizing automorphic function $\varphi(t)$, a proof which he has treated in the second edition of his Funktionentheorie with detail. The prime function is defined in Lecture V as a limit

$$\Omega(t, \tau) = \lim_{\Delta t=0,\ \Delta\tau=0} \sqrt{\Delta t \Delta \tau}\, e^{-\Pi_{t\tau}^{t+\Delta t,\, \tau+\Delta\tau}},$$

where $\Pi_{\mu\nu}^{\xi\eta}$ is a function analogous to the difference

$$P_{\xi\eta}^{xy} = P_{\xi\eta}(x) - P_{\xi\eta}(y)$$

formed for a suitably chosen integral $P_{\xi\eta}(x)$ of the third kind on the Riemann surface. It is seen that the definition involves a function of four arguments, and it is on account of this circumstance that the fifth lecture is a natural continuation of the preceding four. For the theorems of Lecture II, in particular, describing the analytic character of a function of several simultaneous variables resulting from known properties of the function when all but one variable are considered as parameters, play an important and apparently indispensable rôle in establishing the existence and properties of the functions $\Pi$ and $\Omega$.

GILBERT AMES BLISS.

THE UNIVERSITY OF CHICAGO.

---

## SHORTER NOTICE.

*Mathematiker-Anekdoten.* By W. AHRENS. Band XVIII of the Mathematische Bibliothek edited by Lietzmann and Witting. Leipzig, B. G. Teubner, 1916. Pp. 56. Price 80 pfennige.

ANYTHING from the pen of Dr. Ahrens is always sure to be interesting. His Mathematische Unterhaltungen und Spiele, his Mathematische Spiele, and his Scherz und Ernst in der Mathematik are all well known and are everywhere appreciated. He is one of those rare writers on the bizarre in mathematics who keep their balance and turn out works which have the stamp of dignity and learning, while at the same time revealing the lighter side of the science. In other words, he is a man who evidently combines in his own soul the Scherz

und Ernst, as he combines these elements in the title of one of his best known works.

On this account the reader is justified in expecting something worth his attention in a book of Mathematiker-Anekdoten compiled by such a writer; and if, by chance, he is a little disappointed in the diminutive size of the work under review, he may well ask himself what he expects for 80 pfennige. Surely, one cannot look for a seven-course intellectual dinner for 20 cents.

To American and English readers the chief interest in the anecdotes lies in the fact that most of them are not to be found in works in our language. The anecdotes in themselves are not numerous, nor are they, on the whole, as interesting as those to be found in some of Rebière's curious collections, but they are not of the stereotyped variety and hence they are welcome.

The first in Dr. Ahrens's list is Adam Riese, known to most Germans from the phrase "nach Adam Riese," which led, no doubt, to "according to Cocker" in England and "according to Daboll" in this country. The story is not of particular interest, but it lightens up, in a way, the general impression of Riese which comes to one from reading his works. The next one in the list is Gauss, concerning whose school days a couple of interesting stories are told; how he, at the age of nine, saw at a glance how to sum an arithmetic series, and how, at fifteen, he was reading Newton, Euler, and Lagrange.

Among others concerning whom Dr. Ahrens relates various anecdotes are the mathematical prodigy Henri Mondeux, with his examination by Cauchy; Riemann, "einer der genialsten und tiefsinnigsten Mathematiker aller Zeiten;" Grassmann, whom the vocational-guidance extremists would quite certainly have turned from the paths of mathematics in his younger days; Schellbach, whose nature the poet Novalis described, to the mind of Dr. Ahrens, in the statement that "Das Leben der Götter ist Mathematik. Alle göttlichen Gesandten müssen Mathematiker sein. Reine Mathematik ist Religion. Die Mathematiker sind die einzig Glücklichen. Der echte Mathematiker ist Enthusiast per se. Ohne Enthusiasmus keine Mathematik;" Ohm the physicist; August; Lasswitz the poet, who was professor of mathematics in the Gymnasium at Gotha; Euler; and Arago, with his interesting examination by Monge and Legendre. There are also notes

on certain interesting newspaper problems, a problem of exchange under war conditions, an incident in connection with Kästner's epigram on logarithms, and, finally, the Fermat problem.

All this is told in a style that is in harmony with the nature of the work and that would repay the reader even if the anecdotes themselves did not seem worth his time, which is far from being the case.

DAVID EUGENE SMITH.

---

## NOTES.

THE July number (volume 17, number 3) of the *Transactions of the American Mathematical Society* contains the following papers: "On a general class of series of the form $\Sigma c_n g(x+n)$," by R. D. CARMICHAEL; "The geometries associated with a certain system of Cremona groups," by J. W. YOUNG and F. M. MORGAN; "A reduction of certain analytic differential equations to differential equations of an algebraic type," by W. D. MACMILLAN; "A new canonical form of the elliptic integral," by BESSIE I. MILLER; "On the notion of summability for the limit of a function of a continuous variable," by L. L. SILVERMAN; "On the factorization of Cremona plane transformations," by J. W. ALEXANDER; "Weierstrass's non-differentiable function," by G. H. HARDY; "Finite groups represented by special matrices," by G. A. MILLER; "On infinite regions," by W. F. OSGOOD; "Point sets and allied Cremona groups (Part II)," by A. B. COBLE; "Infinite products of analytic matrices," by G. D. BIRKHOFF.

THE July number (volume 38, number 3) of the *American Journal of Mathematics* contains the following papers: "A class of asymptotic orbits in the problem of three bodies," by L. A. H. WARREN; "Some invariants of the ternary quartic," by H. I. THOMSEN; "Functions of surfaces with exceptional points or curves," by C. A. FISCHER; "Dupin's cyclide as a self-dual surface," by MABEL M. YOUNG; "Projective differential geometry of one-parameter families of space curves, and conjugate nets on a curved surface. Second memoir," by G. M. GREEN; "The asymptotic equation and satellite conic of the plane quartic," by TERESA COHEN.

THE June number (series 2, volume 17, number 4) of the *Annals of Mathematics* contains the following papers: "On the Wronskian test for linear dependence," by MAXIME BÔCHER; "Note on a theorem on envelopes," by W. R. LONGLEY; "Non-essential singularities of functions of several complex variables," by DUNHAM JACKSON; "A congruence of circles," by F. W. BEAL; "A case of iteration in several variables," by A. A. BENNETT; "The arithmetic genus of an algebraic manifold immersed in another," by S. LEFSCHETZ; "A characteristic property of self-projective curves," by L. L. DINES.

THE September number (volume 18, number 1) of the *Annals of Mathematics* contains: "On the surface of lowest degree passing through a given curve in space," by T. HAYASHI; "A practical method of determining elementary divisors," by H. T. BURGESS; "Conjugate systems with equal point invariants," by L. P. EISENHART; "On the derivatives of a function at a point," by J. F. RITT; "An existence theorem for the solution of a type of real mixed difference equation," by A. A. BENNETT; "Double elliptic geometry in terms of point and order alone," by J. R. KLINE; "An application of a group of order 16 to a configuration on an elliptic curve," by A. EMCH.

AT the meeting of the Edinburgh mathematical society on June 9 the following papers were read: By E. T. WHITTAKER: "On the solution of Riccati's equation by continued fractions"; by H. DATTA: "On symmetric determinants and Pfaffians"; by A. B. JEFFERY: "Bipolar and toroidal coordinates"; by W. P. MILNE: "The polars of the Hessian of a cubic curve" and "Determinantal systems of points"; by F. G. TAYLOR: "An involution pencil of whole-plane birationally related cubics."

A meeting of mathematicians from Sweden, Denmark, Finland, and Norway was held at Stockholm August 30–September 2. It will be recalled that the Sixth International Congress of Mathematicians was to have met at Stockholm at that time.

THE firm of John Wiley and Sons announces the publication of Lectures on Ten British Mathematicians of the Nineteenth Century, by the late ALEXANDER MACFARLANE.

THE following advanced courses in mathematics are offered at the Italian universities during the academic year 1916–1917:

UNIVERSITY OF BOLOGNA.—By Professor P. BURGATTI: Mechanics of continuous bodies, especially of viscous fluids and elastic bodies, three hours.—By Professor L. DONATI: Modern electromagnetic theories, with reference to recent researches of H. Kamerlingh Onnes and A. Righi, three hours.—By Professor F. ENRIQUES: Theory of singularities of algebraic functions, three hours.—By Professor S. PINCHERLE: Theory of sets; theory of functions of a real variable; the modern meaning of definite integration; Fourier's series, three hours.

UNIVERSITY OF CATANIA.—By Professor E. DANIELE: Introduction to the general theories of electricity and magnetism, four hours.—By Professor G. PENNACCHIETTI: Hydrodynamics, four hours.—By Professor G. SCORZA: Abelian functions with geometrical applications, three hours.—By Professor C. SEVERINI: Advanced part of differential and integral calculus, four hours.

UNIVERSITY OF GENOA.—By Professor E. E. LEVI: Functions of one and several complex variables, four hours.—By Professor G. LORIA: Theory of algebraic surfaces, especially cubics and quartics, three hours.—By Professor O. TEDONE: Principles of the mechanics of electrons, three hours.

UNIVERSITY OF MESSINA.—By Professor P. CALAPSO: Functions of a complex variable; elliptic functions, four hours.—By Professor G. Z. GIAMBELLI: Hyperspaces; moduli of algebraic forms; forms in several series of variables, four hours.—By Professor E. LAURA: Electromagnetic theory of light, three hours.

UNIVERSITY OF NAPLES.—By Professor F. AMODEO: History of mathematics from Galileo to Newton, three hours.—By Professor A. DEL RE: Analysis of Grassmann and theoretical astronomy, four and one half hours.—By Professor R. MARCOLONGO: Mathematical theory of vibrations of isotropic elastic bodies, three hours.—By Professor D. MONTESANO:

Geometry of the space of conics, three hours.—By Professor E. PASCAL: Selected topics of mathematical analysis, three hours.—By Professor L. PINTO: Electro-optics; hertzian waves, three hours.

UNIVERSITY OF PADUA.—By Professor F. D'ARCAIS: Harmonic and polyharmonic functions; analytic functions; integral equations, four hours.—By Professor P. GAZZANIGA: Theory of numbers, four hours.—By Professor T. LEVI-CIVITA: General principles of analytic mechanics; newtonian potential; relativity; Einstein's gravitational statics, four hours.—By Professor G. RICCI: Absolute differential calculus; elasticity, four hours.—By Professor F. SEVERI: Algebraic geometry, four hours.—By Professor A. TONOLO: Partial differential equations of the first order, three hours.—By Professor G. VERONESE: Foundations of geometry, three hours.

UNIVERSITY OF PALERMO.—By Professor G. BAGNERA: Theory of complex variables; elliptic and modular functions, three hours.—By Professor M. DE FRANCHIS: Jacobi's inversion problem and Picard's varieties; hyperelliptic surfaces, three hours.—By Professor M. GEBBIA: Mechanics of continuous media; newtonian potential; hydrostatics and hydrodynamics, four and one half hours.—By ————: Mechanics (advanced part), three hours.

UNIVERSITY OF PAVIA.—By Professor L. BERZOLARI: General theory of algebraic forms with geometrical applications, three hours.—By Professor U. CISOTTI: Ancient and modern analytical methods of mathematical physics and their most conspicuous applications, three hours.—By Professor F. GERBALDI: Functions of a complex variable; elliptic functions, three hours.—By Professor G. VIVANTI: Theory of partial differential equations, three hours.

UNIVERSITY OF PISA.—By Professor E. BERTINI: The geometry on an algebraic curve with transcendental method, three hours.—By Professor L. BIANCHI: Theory of functions of a complex variable; elliptic functions, four and one half hours.—By Professor U. DINI: General theory of series, particularly of divergent ones; analytic representation of

function in real and complex fields, four and one half hours.—By Professor G. A. MAGGI: The potential function and its direct applications; equilibrium and motion of elastic bodies, four and one half hours.—By Professor P. PIZZETTI: Introduction to spherical astronomy and to determination of elliptic orbits; mechanical theory of the shape and of the motion of rotation of the planets, four and one half hours.

UNIVERSITY OF ROME.—By Professor G. BISCONCINI: Geometrical applications of calculus, three hours.—By Professor G. CASTELNUOVO: Differential geometry, three hours.—By Professor U. CRUDELI: Functions of lines, three hours.—By Professor L. SILLA: Kinematics and mechanisms, three hours.—By Professor V. VOLTERRA: Selected problems of mathematical physics, three hours; Hydrodynamics and aerodynamics, three hours.

UNIVERSITY OF TURIN.—By Professor T. BOGGIO: Potential; selected topics of analytical mechanics, three hours.—By Professor G. FUBINI: Cantor's numbers; entire and algebraic entire numbers; theory of numbers and forms with algebraic applications; applications of analysis to the theory of numbers, three hours.—By Professor C. SEGRE: Higher views concerning elementary geometry, three hours.—By Professor C. SOMIGLIANA: Electromagnetism with special regard to propagation phenomena, three hours.

PROFESSOR H. HAHN, of the University of Czernowitz, has been appointed professor of mathematics at the University of Bonn.

PROFESSOR C. JUEL has retired from the editorial board of the *Nyt Tidsskrift for Matematik*. Professor P. HEEGAARD, of the University of Copenhagen, is his successor.

PROFESSOR E. ZERMELO, of the University of Zurich, has retired. Professor R. FUETER, of the technical school at Carlsruhe, has been appointed his successor.

PROFESSOR G. SCORZA, of the University of Parma, has been promoted to a full professorship of projective and descriptive geometry, and transferred to the University of Catania for the current year.

THE Italian society of sciences (the forty) has awarded its gold medal for 1915 to Professor P. CALAPSO, of the University of Messina, for his researches in geometry, especially in the extensional deformation of quadrics.

PROFESSOR F. GERBALDI, of the University of Pavia, has been elected to membership in the Royal institute of Lombardy.

DURING the months of May and June, Professor J. HADAMARD, of the Ecole Polytechnique, gave a series of lectures on mathematical analysis at the Universities of Bologna, Genoa, and Rome.

DR. J. W. L. GLAISHER, of Trinity College, Cambridge, has been elected a member of the Royal society of Edinburgh.

HON. BERTRAND RUSSELL, former fellow of Trinity College, Cambridge, will be unable to lecture at Harvard University during the present year.

AT Vassar College, Dr. ELIZABETH B. COWLEY has been promoted to an associate professorship of mathematics.

DR. C. L. E. MOORE, of the Massachusetts Institute of Technology, has been promoted to an associate professorship of mathematics.

AT the Rice Institute, Professor G. C. EVANS has been promoted to a full professorship of pure mathematics, and Dr. W. C. GRAUSTEIN to an assistant professorship of mathematics. Dr. F. D. MURNAGHAN, of Johns Hopkins, has been appointed to an instructorship in mathematics, and Mr. H. E. BRAY, of Harvard University, has been appointed to one of the two graduate fellowships in mathematics.

DR. A. A. BENNETT, of Princeton University, has been appointed adjunct professor of mathematics at the University of Texas.

PROFESSOR C. N. HASKINS, of Dartmouth College, has been promoted to a full professorship of mathematics.

MR. W. C. EELLS has resigned his position in the department of mathematics at the United States Naval Academy to accept an appointment as professor of applied mathematics at Whitman College, Walla Walla, Wash.

DR. D. D. LEIB, instructor in mathematics at the Sheffield Scientific School, Yale University, has been appointed assistant professor of mathematics and physics at the Connecticut College for Women.

AT the University of Kansas, Dr. S. LEFSCHETZ and Mr. J. J. WHEELER have been promoted to assistant professorships and Mr. E. B. MILLER, of the University of Chicago, has been appointed to an instructorship in mathematics.

AT Northwestern University, Dr. C. N. YEATON has been promoted to an assistant professorship of mathematics. Dr. C. E. WILDER, of Pennsylvania State College, Dr. F. E. WOOD, and Mr. I. ROMAN have been appointed instructors in mathematics.

DR. OTTO DUNKEL, of the University of Missouri, has been appointed assistant professor of mathematics at Washington University, St. Louis.

DR. J. E. ROWE, of Pennsylvania State College, has been promoted from an assistant to an associate professorship of mathematics.

DR. R. L. BORGER, of the University of Illinois, has been appointed professor of mathematics at Ohio University, Athens, Ohio.

DR. G. H. LIGHT has been appointed instructor in mathematics at the University of Colorado.

DR. J. V. DE PORTE, of Cornell University, has been appointed instructor in mathematics at the State college for teachers, Albany.

AT Syracuse University, Dr. J. L. JONES, of the University of Pittsburgh, has been appointed instructor in mathematics.

At the University of Nebraska, Mr. A. Babbett and Miss M. A. Colpitts have been appointed instructors in mathematics.

Dr. N. Altshiller, of the University of Colorado, has been appointed instructor in mathematics at the University of Oklahoma.

Dr. S. P. Sousley, of St. Mary's College, Emmetsburg, Maryland, has been appointed instructor in mathematics at Pennsylvania State College.

Professor E. Mach, of the University of Vienna, died February 22, 1916, at the age of 78 years.

Dr. K. Schwarzschild, director of the observatory at Potsdam, died May 11, 1916, at the age of 42 years.

Professor W. Vogt, of the University of Heidelberg, was killed in battle March 3, 1916, at the age of 32 years.

Don Jose Echegaray, professor of mathematical physics in the University of Madrid, and distinguished also as a poet and dramatic author, died September 15, 1916, at the age of eighty-three years. Professor Echegaray was a member of the academy of science of Madrid, which celebrated the fiftieth anniversary of his admission to the academy on March 12, 1916.

Edgar H. Harper, professor of mathematical physics in University College, Cork, known for his work on aviation, was killed while serving as a lieutenant.

Dr. G. A. Hill, formerly professor of physics at Harvard University, and author of a number of text books in physics and mathematics, died August 17, 1916, at the age of 74 years.

Professor J. P. D. John, formerly professor of mathematics at De Pauw University, died August 7, 1916, at the age of 73 years.

Professor W. C. Esty, of Amherst College, died July 27, 1916, at the age of 78 years. Professor Esty was Walker

professor of mathematics from 1865 to his retirement from active service in 1905.

DR. EMORY MCCLINTOCK died July 10 at the age of seventy-six years. Dr. McClintock was the first president of the American Mathematical Society, serving in that office from 1890 to 1894. He graduated from Columbia College in 1859 and was assistant professor of mathematics at Columbia in 1859 and 1860. He was actuary of the Asbury Life Insurance Company from 1867 to 1871, and of the Northwestern Mutual Life Insurance Company from 1871 to 1888, when he became connected with the Mutual Life Insurance Company of New York, of which he was also vice-president from 1905 to his retirement in 1911. He was a frequent contributor to the *American Journal of Mathematics* and other journals, and was one of the founders of the *Transactions* of the American Mathematical Society.

FREDERICK W. FRANKLAND died in New York City on July 24, in the sixty-second year of his age. He was a Fellow of the Institute of Actuaries, and a member of the American Mathematical Society since 1893. He was for several years government examiner in mathematics and insurance in New Zealand, and was author of numerous papers on mathematical, metaphysical, and sociological subjects.

JOSIAH ROYCE, Alvord professor of philosophy at Harvard University, died September 14, 1916, at the age of sixty years.

---

## NEW PUBLICATIONS.

### I. HIGHER MATHEMATICS.

ABENBÉDER. See PÉREZ (J. A. S.).

BENEDICT (S. R.). A comparative study of the early treatises introducing into Europe the Hindu art of reckoning. (Diss., University of Michigan.) Concord, N. H., Rumford Press, 1916. 6+126 pp.

BIANCHI (L.). Lezioni sulla teoria delle funzioni analytiche di variabile complessa e delle funzioni ellittiche. 2a edizione. Pisa, E. Spoerri (F. Mariotti), 1916. 8vo. 685 pp. L. 25.00

CASHMORE (M.). Fermat's last theorem. London, Bell, 1916. 63 pp. 2s.

COOLIDGE (J. L.). A treatise on the circle and the sphere. Oxford, Clarendon Press, 1916. 602 pp. 21s.

GREGORY (R. A.). Discovery: or, the spirit and service of science London, Macmillan, 1916. 10+340 pp. 5s.

JESSOP (C. M.). Quartic surfaces with singular points. Cambridge, University Press, 1916. 35+197 pp. 12s.

LEUDET (M.). See PETIT (G.).

LÖVGREN (B. N.). Om hela tals delbahet med hela tal. Stockholm, A. Bonniers Verlag, 1915. 30 pp. Kr. 1.50

MACFARLANE (A.). Lectures on ten British mathematicians of the nineteenth century. New York, Wiley, 1916. 8vo. 148 pp. Cloth. $1.25

PÉREZ (J. A. S.). Compendio de álgebra de Abenbéder. Texto árabe, traducción y estudio. Madrid, E. Maestre, 1916. 8vo. 48+117+78 pp. Pes. 6.

PETIT (G.) et LEUDET (M.). Les allemands et la science. Paris, Alcan, 1916. 20+375 pp. Fr. 3.50

SÖDERBLUM (A.). Inledning till rymdens analytiska geometri. Gotenburg, Gumperts Verlag, 1915. 65 pp. Kr. 2.50

TOGLIATTI (E. G.). Le superficie di sesto ordine con infinite coniche. Milano, Hoepli, 1916. 4to. 64 pp. L. 3.85

VOLTERRA (V.). Teoria delle potenze dei logaritmi e delle funzioni di composizione. Roma, tip. R. Accademia dei Lincei, 1916. 4to. Pp. 167–249.

ZORETTI (L.). Exercices numériques et graphiques de mathématiques sur les Leçons de mathématiques générales du même auteur. Paris, Gauthier-Villars, 1916. 8vo. 16+124 pp. Fr. 7.00

## II. ELEMENTARY MATHEMATICS.

CLAPHAM (C. B.). Arithmetic for engineers, including simple algebra, mensuration, logarithms, graphs, and the slide rule. London, Chapman and Hall, 1916. 11+436 pp. 5s. 6d.

DICKNETHER (F.). Rechenbuch für höhere Lehranstalten: 1ter Teil, 5te und 6te Auflage; 2ter Teil, 5te Auflage; 3ter Teil, 4te und 5te Auflage. München, J. Lindauer. 120+142+160 pp. M. 1.70+1.70+1.90.

FENKNER (H.). Mathematisches Uebungsbuch. 2ter Teil (Pensum der Obersekunda und Prima). Ausgabe A (für Gymnasien); Ausgabe B (für Realgymnasien und Oberrealschulen). Berlin, Salle, 1915. 363+396 pp. M. 3.40+3.60

HUDSON (H. P.). Ruler and compasses. (Modern Mathematical Series.) London, Longmans, 1916. 6+143 pp. 6s.

MAXSON (W. S.). Fundamental number-work method; a logical, practical and systematic method for acquiring thorough knowledge of the foundation processes of fundamental number. Yonkers, N. Y., Self-keyed Number Work Company, 1916. 220 pp. $2.00

MONROE (W. S.). Report of the use of the Courtis standard research tests in arithmetic in twenty-four cities. Emporia, Kansas State Normal School, 1915. 8vo. 94 pp.

PENDLEBURY (C.). Revision papers in arithmetic. London, Bell, 1916. 15+68+18 pp. 1s.

ROBERTSON (J. W.). A shilling arithmetic. London, Bell, 1916. 8+191 pp. 1s.

TALLANT (G. A.). See UMHOLTZ (F. H.).

TALLANT (G. A.) and UMHOLTZ (F. H.). Common school advanced arithmetic. Chicago, Laird and Lee, 1916. 8vo. 6+384 pp. $0.55

UMHOLTZ (F. H.). See TALLANT (G. A.).

UMHOLTZ (F. H.) and TALLANT (G. A.). Common school elementary arithmetic. Chicago, Laird and Lee, 1916. 7+280 pp. $0.35

## III. APPLIED MATHEMATICS.

ANNUARIO astronomico pel 1917 publicato dal r. osservatorio di Pino Torinese. Torino, tip. s. Giuseppe degli Artigianelli, 1916. 8vo. 132 pp.

ANNUARIO publicado pelo observatorio nacional do Rio de Janeiro, 1916. Rio de Janeiro, 1916. 6+259 pp.

BARBILLION (L.). Leçons sur le fonctionnement des groupes électrogènes en régime troublé. Paris, Gauthier-Villars, 1916. 2+306 pp. Fr. 11.00

COMINOTTO (E.). Compasso cubico. Piacenza, tip. V. Porta, 1915. 8vo. 10 pp.

DUHEM P.). Le système du monde. Histoire des doctrines cosmologiques de Platon à Copernic. Tomes 2–3. Paris, Hermann, 1914–1915. 522+549 pp.

DUNK (J. L.). Hyperacoustics. Division 1: Simultaneous tonality. London, Dent, 1916. 6+311 pp. 7s. 6d.

HOUSTOUN (R. A.). A treatise on light. London, Longmans, 1915. 11+478 pp. 7s. 6d.

MILLOSEVICH (E.). I pianetini (654) Zelinda e (303) Josephina; Elementi parabolici ed efemeride della cometa 1915 *a* (Mellish); Perturbazioni sulle coordinate per l'azione di Terra a Giove (r. osservatorio astronomico al Collegio Romano). Roma, tip. Unione ed., 1916. 4to. 14 pp.

NAYLER (W. A.). See TUCKEY (C. O.).

ROMEO (A.). Compendio di topografia. Torino, Casanova (Olivero e C.), 1916. 93 pp. con 6 tavole.

ROTIGLIANO (S.). Costruzioni di strade e gallerie. Milano, Hoepli, 1916. 23+808 pp. L. 18.00

RUSSELL (A.). A treatise on the theory of alternating currents. Vol. 2, 2d edition. Cambridge, University Press, 1916. 19+340 pp. 15s.

SEGRETI del giuco del lotto matematicamente studiati e rivelati. Firenze, Emporio poligrafico (stab. tip. Aldino), 1916. 16mo. 14 pp. con prospetto. L. 2.50

TUCKEY (C. O.) and NAYLER (W. A.). Statics: a first course. Oxford, Clarendon Press, 1916. Cr. 8vo. 299 pp. 3s. 6d.

VINASSA DE REGNY (P.). La conquista della forza: discorso letto per la solenne inaugurazione degli studi nella r. Università di Parma il giorno 8 novembre 1915. Parma, tip. E. Pelati, 1916. 8vo. 22 pp.

WEREIDE (T.). Statistical theory of energy and matter. Kristiania, Gyldendalske Boghandel Nordisk Forlag, 1915. 16+170 pp.